AS and A-level Mathematics

Teaching Guidance

AS 7356 and A-level 7357

For teaching from September 2017
For AS and A-level exams from June 2018

Version 3.0 September 2019

Our specification is published on our website (aqa.org.uk). We will let centres know in writing about any changes to the specification. We will also publish changes on our website. The definitive version of our specification will always be the one on our website and may differ from printed versions.

You can download a copy of this teaching guidance from our All About Maths website (allaboutmaths.aqa.org.uk/). This is where you will find the most up-to-date version, as well as information on version control.

Contents

Key

Pure maths

Mechanics

Statistics

General information – disclaimer

This AS and A-level Mathematics teaching guidance will help you plan your teaching by further explaining how we have interpreted content of the specification and providing examples of how the content of the specification may be assessed. The teaching guidance notes do not always cover the whole content statement.

The examples included in this guidance have been chosen to illustrate the level at which this content will be assessed. The wording and format used in this guidance do not always represent how questions would appear in a question paper. Not all questions in this guidance have been through the same rigorous checking process as the ones used in our question papers.

Several questions have been taken from legacy specifications and therefore contain a higher proportion of marks for AO1 than will be found in a suite of exam papers for the new AS and A-level Mathematics specifications.

This guidance is not, in any way, intended to restrict what can be assessed in the question papers based on the specification. Questions will be set in a variety of formats including both familiar and unfamiliar contexts.

All knowledge from the GCSE Mathematics specification is assumed.

Subject content

The subject content for AS and A-level Mathematics is set out by the Department for Education (DfE) and is common across all exam boards.

This document is designed to illustrate the detail within the content defined by the DfE.

Content in **bold type** is contained within the AS Mathematics qualification as well as the A-level Mathematics qualification. Content in standard type is contained only within the A-level Mathematics qualification.

Synopticity and problem solving

One of the requirements of the new A-level specification is to test the content synoptically and for students to apply the knowledge they have in unfamiliar areas. In section four of the specification it says that students should be able to ‘draw together information from different areas of the specification’ and ‘apply their knowledge and understanding in practical and theoretical contexts’.

The effect of these requirements is that, in order to assess a student’s ability to bring together the different elements of their mathematical knowledge, we need to allocate a certain number of marks to problems which are, unpredictable. We can ask students to use any techniques in the specification to solve problems. Teachers are advised to give students opportunities to solve unfamiliar problems based on well embedded knowledge throughout the course so that when faced with unconventional questions in the exam they are prepared to attempt it.

We are conscious that synoptic questions can be more demanding than others. We have to include one synoptic question on each exam paper, but we are very careful to make questions as accessible as possible.

It can often be the case that ‘problems’ are more demanding than routine questions, but as we need to allocate 25% (20% at AS) of marks to problem solving and modelling assessment objectives, we work hard to make these questions as accessible as we can.

There is strong research evidence that students are best placed to solve problems when they have expert knowledge of the maths required. For teachers, this suggests that setting demanding problems on topics as you teach them is not likely to be effective in developing a positive approach to solving problems. Beginning the course with an element of problem solving based on well understood GCSE Maths is more likely to be effective.

Calculators

Using calculators in exams is more important now than it was in the previous modular specification and we really embrace this development.

Here's a list of some of the more common things we expect students to be able to do with a calculator in exams:

- solve quadratic equations
- find the coordinates of the vertex of a quadratic function
- solve quadratic inequalities
- solve simultaneous linear equations in two variables
- calculate summary statistics for a frequency distribution
- repeat an iterative process, including the Newton-Raphson method
- find binomial and normal probabilities and find the z-value for a normal distribution
- calculate a definite integral
- calculate the derivative of a function at a given point
- solve equations when the question permits, eg when problem solving and modelling.

Generally, if a calculator can be used then we expect students to do so and there will be no extra credit for using a handwritten method. The specification says that students should ''use technology such as calculators and computers effectively and recognise when such use may be inappropriate'.' Consequently, students should be aware of the range of functions available on their calculator as well as their limitations.

Calculators will not be appropriate in every situation.

- We will include parameters in some questions so that they cannot be completed on a calculator when we need to test students' abilities to carry out particular techniques.
- 'Exact value' means that the answer will often involve surds, e, or π and should not be given in decimal form.
- The instructions 'Show that', 'Prove' and 'Fully justify' mean we require a method with all steps clearly shown. Calculators may be used to help with intermediate steps but the input and output should be written down and the working must meet the requirements of the question.

Mark scheme

We've redesigned our mark schemes to help you and your students fully understand what's expected in order to gain marks. The mark scheme is designed to be applied positively and to reward students for making progress even if they have not always completed each step correctly. In our mark schemes you will find this instruction to markers:

> *When a choice of answers and/or methods is given and the student has not clearly indicated which answer they want to be marked, mark positively, awarding marks for all of the student's best attempts. Withhold marks for final accuracy and conclusions if there are conflicting complete answers or when an incorrect solution (or part thereof) is referred to in the final answer.*

You will often find marks come in pairs, with a method mark followed by an accuracy mark. For example:

Marking Instruction	**AO**	**Marks**
Differentiates (at least one term correct)	AO1.1a	M1
Correctly differentiates	AO1.1b	A1

The first mark is a method mark given for knowing that the correct next step is to differentiate and is given for knowing what to do, even if the maths is not fully correct. The second mark is an accuracy mark and is given for knowing what to do and doing it correctly. The method mark, if given, will follow through on longer questions.

Here are the important features:

- our marking instructions focus on the mathematical principles being assessed, without being over-prescriptive of the techniques or methods to use
- where marks are awarded for accuracy (A), a high proportion allow 'follow through' (ft). This means students can still receive credit after an incorrect result if the next step has been completed successfully.
- a high proportion of marks are 'method' marks (M).
- the 'typical solution' on the right-hand side of the mark scheme shows what a very good solution could look like, but doesn't describe what students must do. If a student has used another method, sensible application of the mark scheme should be possible to give appropriate credit.
- reasoning (R) marks are awarded for communicating mathematically and applying mathematical principles. They could be given for correct and fully justified work, or for reasoning what the next step needs to be in solving a problem.
- explanation (E) marks are for saying what it is that they are doing or thinking
- you'll see how the assessment objectives are assessed, with each mark allocated an AO
- we won't always need a particular method to be used in a student's solution for them to gain marks, however we also respect that some parts of the specification require a certain method to be used and we have to assess this specifically.

A Proof

A1	Understand and use the structure of mathematical proof, proceeding from given assumptions through a series of logical steps to a conclusion; use methods of proof, including proof by deduction, proof by exhaustion. Disproof by counter example. Proof by contradiction (including proof of the irrationality of $\sqrt{2}$ and the infinity of primes, and application to unfamiliar proofs).

Assessed at AS and A-level

Teaching guidance

Students should be able to:

- set out a clear proof with the correct use of symbols, such as $=, \Rightarrow, \Leftarrow, \Leftrightarrow, \equiv, \therefore, \because$
- understand that considering examples can be useful in looking for structure, but this does not constitute a proof.

Notes

- At A-level 25% (20% at AS) of the assessment material must come from Assessment Objective 2 (reason, interpret and communicate mathematically). A focus on clear reasoning, mathematical argument and proof using precise mathematical language and notation should underpin the teaching of this specification. Students should become familiar with the mathematical notation found in Appendix A of the specification.
- It will not be essential to use any particular notation when writing answers to exam questions, but some questions could assess understanding of this notation.
- Students should understand the sets of numbers $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$

Examples

1 Prove that 113 is a prime number.

2 Prove that for any positive whole number, n, the value of $n^3 - n$ is always a multiple of 3.

3 "For any positive whole number, n, the value of $2n^2 + 11$ is a prime number."

Find a value of n that disproves this statement.

Only assessed at A-level

Examples

1 Assuming $\sqrt{2}$ is a rational number we can write $\sqrt{2} = \frac{a}{b}$, where a and b are positive whole numbers with no common factors.

(a) Show that a must be even.

(b) Show that b must be even.

(c) Using parts (a) and (b), explain why there is a contradiction and state what conclusion can be made about $\sqrt{2}$ as a result.

AQA

B Algebra and functions

B1 Understand and use the laws of indices for all rational exponents.

Assessed at AS and A-level

Teaching guidance

Students should be able to:

- understand and use the following laws:

$$x^a \times x^b = x^{a+b} \qquad x^a \div x^b = x^{a-b} \qquad \left(x^a\right)^b = x^{ab}$$

$$x^{-a} = \frac{1}{x^a} \qquad x^{\frac{a}{b}} = \sqrt[b]{x^a} = \left(\sqrt[b]{x}\right)^a$$

- apply these laws when solving problems in other contexts, for example simplification of expressions before integrating/differentiating, solving equations or transforming graphs.

Examples

1 (a) Write down the values of p, q and r, given that:

(i) $a^2 = \left(a^3\right)^p$

(ii) $\dfrac{1}{\left(b^2\right)^3} = \left(b^3\right)^q$

(iii) $\sqrt{c^3} = \left(\dfrac{1}{c^2}\right)^r$

(b) Find the value of x for which

$$\frac{\left(a^3\right)^x}{a^{\frac{1}{2}}} = \frac{1}{a^2}$$

2 Find $\int \left(x + 1 + \frac{4}{x^2} \right) \mathrm{d}x$

3 A curve has equation $y = \frac{1}{3x^2} + 4x$

Find $\frac{\mathrm{d}y}{\mathrm{d}x}$

4 Find $\int \left(\sqrt[3]{x^4} - 1 \right)^2 \mathrm{d}x$

B2 Use and manipulate surds, including rationalising the denominator.

Assessed at AS and A-level

Teaching guidance

Students should be able to:

- demonstrate they understand how to manipulate surds and rationalise denominators
- answer algebraic questions.

Notes

- Questions requiring simplification of surds should usually be done with a calculator, but students are expected to know the process of rationalising a surd denominator and to be able to show this step.

Examples

1 Show that $\dfrac{\sqrt{25k}-\sqrt{9k}}{\sqrt{k}}$ where k is an integer, is also an integer.

2 Express $\dfrac{3+\sqrt{a}}{2-\sqrt{a}}$, where a is a positive integer, in the form $m\sqrt{a}+n$

3 A rectangle has length $\left(9+5\sqrt{3}\right)$ cm and area $\left(a+7\sqrt{3}\right)$ cm^2

The width of the rectangle is $\left(m-2\sqrt{3}\right)$, where m and n are integers.

Find the value of a.

4 Show that $\dfrac{5-\sqrt{12}}{4+\sqrt{3}}$ can be written in the form $p+q\sqrt{3}$, where $p, q \in \mathbb{Z}$. Fully justify your answer.

B3

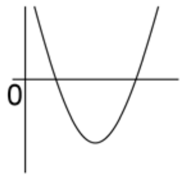
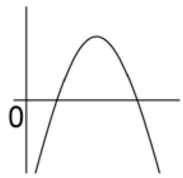
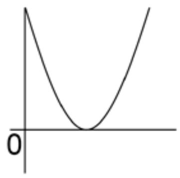
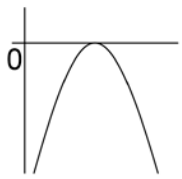
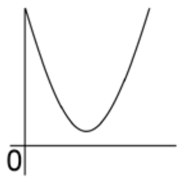
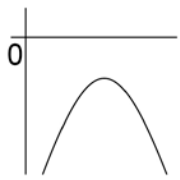
Work with quadratic functions and their graphs; the discriminant of a quadratic function, including the conditions for real and repeated roots; completing the square; solution of quadratic equations including solving quadratic equations in a function of the unknown.

Assessed at AS and A-level

Teaching guidance

Students should:

- be able to sketch graphs of quadratics, ie of $y = ax^2 + bx + c$
- be able to identify features of the graph such as points where the graph crosses the axes, lines of symmetry or the vertex of the graph
- know and use the following:

$b^2 - 4ac > 0$	Distinct real roots		
$b^2 - 4ac = 0$	Equal roots		
$b^2 - 4ac < 0$	No real roots		

Note: a quadratic described as having real roots will be such that $b^2 - 4ac \geq 0$

- be able to complete the square and use the resulting expression to make deductions, such as the maximum/minimum value of a quadratic or the number of roots

  Note: We expect students to use a calculator to solve quadratic equations and to find the coordinates of the vertex. There is no need for substitution in the quadratic formula or completing the square to justify solutions. We expect an understanding of quadratic functions, but the routine solution of equations is not in itself part of this understanding.

Teaching guidance continued

- be able to solve quadratic equations in a function of the unknown, where the function may be, for example, trigonometric or exponential.

Note: quadratic equations may arise from problems set in a variety of contexts taken from mechanics and statistics.

Examples

1 Express $3x^2 + 8x + 19$ in the form $a(x + p)^2 + q$, where a, p and q are integers.

2 Find the values of k for which the equation $x^2 - 2(k + 1)x + 2k^2 - 7 = 0$ has equal roots.

3 Show that $x = 0$ is a solution of $9^x - 3^{x+1} + 2 = 0$ and find the other solution, giving it in the form $\dfrac{\ln a}{\ln b}$ where a and b are integers.

4 Given that $$\frac{3 + \sin^2\theta}{\cos\theta - 2} = 3\cos\theta$$

show that $$\cos\theta = -\frac{1}{2}$$

5 Show that $3^{2x} - 3^{x+1} - 4 = 0$

has exactly one solution, giving this solution in an exact form. Fully justify your answer.

B4

Solve simultaneous equations in two variables by elimination and by substitution, including one linear and one quadratic equation.

Assessed at AS and A-level

Teaching guidance

Students should be able to:

- understand the relationship between the algebraic solution of simultaneous equations and the points of intersection of the corresponding graphs
- in the case of one linear and one quadratic equation, recognise the geometrical significance of the discriminant of the resulting quadratic
- solve a pair of linear simultaneous equations using a calculator.

Notes

- Simultaneous equations could arise from problems set on a variety of topics including mechanics and statistics.
- In situations where simultaneous linear equations have to be solved to give numerical solutions, we expect students to do so with a calculator. No working is required. However, there could still be situations where coefficients are given as exact values, or as parameters, when algebraic methods will need to be used.

Examples

1 The straight line L has equation $y = 3x - 1$

The curve C has equation $y = (x + 3)(x - 1)$

(a) Sketch on the same axes the line L and the curve C, showing the values of the intercepts on the x-axis and y-axis.

(b) Show that the x-coordinates of the points of intersection of L and C satisfy the equation $x^2 - x - 2 = 0$

(c) Hence find the coordinates of the points of intersection of L and C

2 The curve C has equation $y = k(x^2 + 3)$, where k is a constant.

The line L has equation $y = 2x + 2$

Show that the x-coordinates of any points of intersection of the curve C with the line L satisfy the equation $kx^2 - 2x + 3k - 2 = 0$

3 A circle has equation

$$x^2 + y^2 - 10y + 20 = 0$$

A line has equation

$$y = 2x$$

(a) Show that the x-coordinate of any point of intersection of the line and the circle satisfies the equation $x^2 - 4x + 4 = 0$

(b) Hence, show that the line is a tangent to the circle and find the coordinates of the point where it touches the circle

4 The first term of an arithmetic series is 1
The common difference of the series is 6

(a) Find the 10th term of the series.

(b) The sum of the first n terms of the series is 7400

(i) Show that $3n^2 - 2n - 7400 = 0$

(ii) Find the value of n

Note: part (b) features content that is A-level only, and would not be asked at AS.

B5

Solve linear and quadratic inequalities in a single variable and interpret such inequalities graphically, including inequalities with brackets and fractions.

Express solutions through correct use of 'and' and 'or', or through set notation.

Represent linear and quadratic inequalities such as $y > x + 1$ and $y > ax^2 + bx + c$ graphically.

Assessed at AS and A-level

Teaching guidance

Students should be able to:

- give the range of values which satisfy more than one inequality
- illustrate regions on sketched graphs, defined by inequalities
- define algebraically inequalities that are given graphically.

Notes

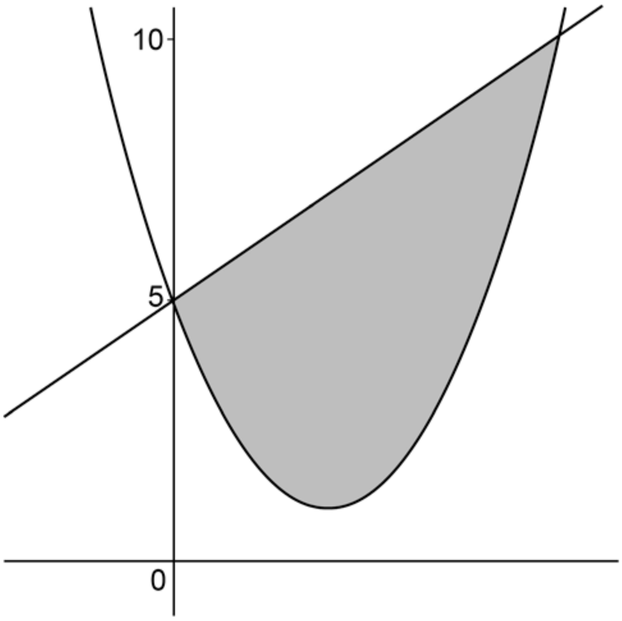
- Dotted/dashed lines or curves will be used to indicate strict inequalities.
- Overarching theme 1.3 is of particular relevance here. Students are required to demonstrate an understanding of and use the notation (language and symbols) associated with set theory (as set out in Appendix A of the specification). Students may be required to apply this notation to the solutions of inequalities.
- There are many ways of representing solution sets using set notation. Typically the variable used is the same as that used in the question, but any letter could be used. We would advise using x or y whenever possible.
- Students are expected to understand notation such as:
    - $\{x : x < 2\}$ or $(-\infty, 2)$
    - $\{x : x \le -1\} \cup \{x : x \ge 2\}$ or $(-\infty, -1] \cup [2, \infty)$
- There is no need to state $x \in \mathbb{R}$, because we assume we are using real numbers.
- If a question requires an answer to be written in set notation we would accept any mathematically correct notation; we would accept the word ‘or’ instead of ‘$\cup$’ Our intention is always to reward correct mathematics.
- Questions will always say how a required region should be indicated eg shade and label the region R.

Examples

1 Find the values of k which satisfy the inequality $3k^2 - 2k - 1 < 0$

Note: This question does not ask for the solution to be given in any particular form. Set notation would not be required.

2 The diagram shows the graphs of $y = x + 5$ and $y = x^2 - 4x + 5$



State which pair of inequalities defines the shaded region.

Circle your answer.

$y < x+5$ and $y < x^2 - 4x + 5$	$y \le x+5$ or $y > x^2 - 4x + 5$	$y \le x+5$ and $y \ge x^2 - 4x + 5$	$y \ge x+5$ or $y < x^2 - 4x + 5$

3 Find the values of x which satisfy both $x^2 + 2x > 8$ and $3(2x+1) \le 15$

Give your answer using set notation.

B6

Manipulate polynomials algebraically, including expanding brackets and collecting like terms, factorisation and simple algebraic division; use of the factor theorem.

Simplify rational expressions including by factorising and cancelling, and algebraic division (by linear expressions only).

Assessed at AS and A-level

Teaching guidance

Students should be able to:

- manipulate polynomials, which may be embedded in questions focused on other topics
- understand factorisation and division applied to a quadratic or a cubic polynomial divided by a linear term of the form $(x + a)$ where a is an integer.

Notes

- Any correct method will be accepted, eg by inspection, by equating coefficients or by formal division.
- The greatest level of difficulty is exemplified by $x^3 - 5x^2 + 7x - 3$, ie a cubic always with a factor $(x + a)$, where a is a small integer and including the cases of three distinct linear factors, repeated linear factors or a quadratic factor which cannot be factorised.
- The use of a calculator to find roots of a cubic polynomial allows students to identify linear factors using the factor theorem, although this isn't a required technique because such factors might not be of the form $(x + a)$. It would be an acceptable technique in an exam.

Examples

1 Find $\int (2x+1)(x^2 - x + 2)\,dx$

2 The polynomial $p(x)$ is given by $p(x) = x^3 + 7x^2 + 7x - 15$

(a) Prove that $x + 3$ is a factor of $p(x)$

(b) Express $p(x)$ as the product of three linear factors.

3 The polynomial $p(x)$ is given by $p(x) = x^3 + x - 10$

(a) Use the factor theorem to show that $x - 2$ is a factor of $p(x)$

(b) Express $p(x)$ in the form $(x - 2)(x^2 + ax + b)$, where a and b are constants.

Only assessed at A-level

Teaching guidance

Students should be able to:

- understand the factor theorem where the divisor is of the form $(ax + b)$
- simplify rational expressions
- carry out algebraic division where the divisor is of the form $(ax + b)$.

Notes

- Any correct method will be accepted, eg by inspection, by equating coefficients or by formal division.

Examples

1 Express $\dfrac{3x^3 + 8x^2 - 3x - 5}{3x - 1}$ in the form $ax^2 + bx + \dfrac{c}{3x - 1}$, where a, b and c are integers.

2 $f(x) = 4x^3 - 7x - 3$

(a) Use the factor theorem to show that $2x + 1$ is a factor of $f(x)$

(b) Simplify $\dfrac{4x^3 - 7x - 3}{2x^2 + 3x + 1}$

B7

Understand and use graphs of functions; sketch curves defined by simple equations including polynomials, the modulus of a linear function, $y = \frac{a}{x}$ **and** $y = \frac{a}{x^2}$ **(including their vertical and horizontal asymptotes); interpret algebraic solution of equations graphically; use intersection points of graphs to solve equations.**

Understand and use proportional relationships and their graphs.

Assessed at AS and A-level

Teaching guidance

Students should be able to:

- understand, use and sketch straight-line graphs (including vertical and horizontal)
- understand and use polynomials up to cubic (including sketching curves)
- understand and use cubic polynomials with at least one linear factor.
- distinguish between the various possibilities for graphs of cubic polynomials indicating where graphs meet coordinate axes
- understand and use graphs of the functions $y = \frac{a}{x}$ and $y = \frac{a}{x^2}$,
  as well as simple transformations of these graphs (including sketching curves).
- use the following:

Proportionality ($\propto$)	**Equation**
y is proportional to x	$y = kx$
y is proportional to x^n	$y = kx^n$
y is inversely proportional to x^n	$y = \frac{k}{x^n}$

Notes

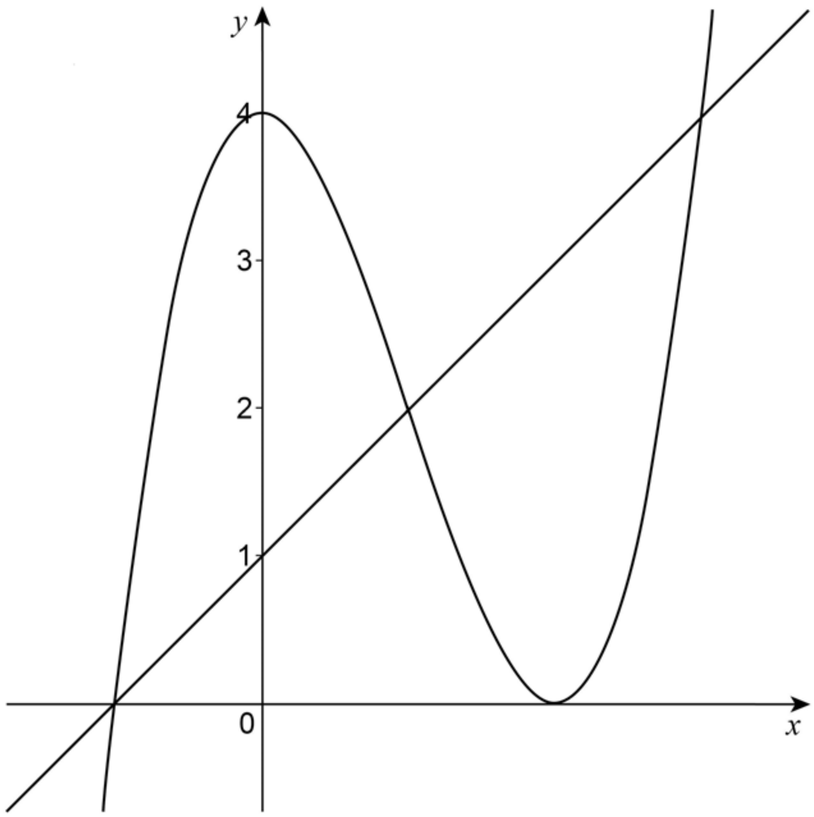
- Sketches should show roots and the y-intercept, provided these are easy to determine.
- Asymptotes should be drawn as dashed lines.

Examples

1 (a) Sketch the graph of $y = \dfrac{1}{x+2}$, clearly labelling the vertical asymptote.

(b) State the transformation which maps the graph of $y = \dfrac{1}{x+2}$ onto the graph of $y = \dfrac{1}{x}$

2 The graphs of $y = x^3 - ax^2 + b$ and $y = cx + d$ are shown on the diagram.



One of the points of intersection of the two graphs is (3, 4)

Find the values of a, b, c and d.

3 P is inversely proportional to the square of Q

$0 < Q < 10$

(a) Sketch the graph of P against Q

(b) The value of P increases by 1.3%

Find the percentage change in the value of Q

Only assessed at A-level

Teaching guidance

Students should:

- know and be able to use:

$$|x| = \begin{cases} x & \text{for } x \geq 0 \\ -x & \text{for } x < 0 \end{cases}$$

- understand and be able to use the graph of $y = |x|$ and combinations of simple transformations of this graph, points of intersection and solutions of equations and inequalities.

Notes

- Functions used will therefore be of the form $y = m|x| + c$ or $y = |mx + c|$

Examples

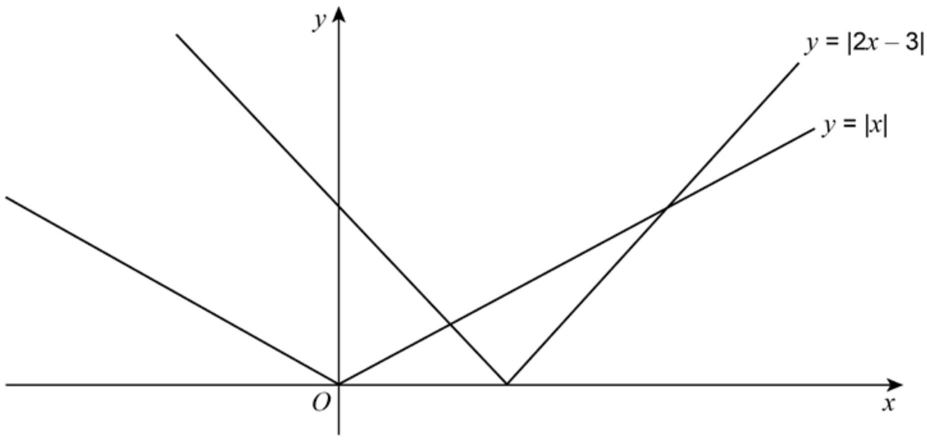
1 (a) Sketch the graph of $y = 4 - |2x|$

(b) Solve $4 - |2x| > x$

2 (a) Sketch the graph of $y = |8 - 2x|$

(b) Solve $|8 - 2x| > 4$

3 The diagram below shows the graphs of $y = |2x - 3|$ and $y = |x|$



(a) Find the exact coordinates of the points of intersection of the graphs of $y = |2x - 3|$ and $y = |x|$

(b) Hence, or otherwise, solve the inequality $|2x - 3| \geq |x|$

B8	Understand and use composite functions, inverse functions, and their graphs.

Only assessed at A-level

Teaching guidance

Students should:

- be able to define a function as a one-to-one or a many-to-one mapping, including the range and domain (co-domain not required)
- understand that the domain and range of a function are sets
- understand and be able to use correct language and notation to describe functions accurately
- know the conditions for the existence of the inverse of a function and the relationship between the domain and range of a function and those of its inverse
- understand that f^{-1} is the inverse of f if and only if $\mathrm{f}^{-1}\mathrm{f}(x) = x$ for all x
- recognise and be able to use notation such as:
    - $\mathrm{f} : x \mapsto 3x^3 - 2x + 4$
    - $\mathrm{f}(x) = x^2$
    - f^{-1} to indicate inverse
- understand that the graph of an inverse function can be found by reflecting in the line $y = x$
- understand the composition of functions:
    - $\mathrm{fg}(x)$ is $\mathrm{f}(\mathrm{g}(x))$
    - and know that the range of g must be a subset of the domain of f in order for fg to be defined.

Notes

- There are many ways of representing the domain and range sets using set notation. Typically the variable used is the same as that used in the question, but any letter could be used. We would advise using x or y whenever possible.
- Students are expected to understand notation such as:
    - $\{x : x < 2\}$ or $(-\infty, 2)$
    - $\{x : x \leq -1\} \cup \{x : x \geq 2\}$ or $(-\infty, -1] \cup [2, \infty)$
- There is no need to state $x \in \mathbb{R}$, because we assume we are using real numbers.
- If a question requires an answer to be written in set notation we would accept any mathematically correct notation; we would accept the word 'or' instead of '$\cup$' Our intention is always to reward correct mathematics.

Examples

1 The functions f and g are defined by

$\mathrm{f}(x) = \mathrm{e}^{2x} - 3$, for all real values of x

$\mathrm{g}(x) = \dfrac{1}{3x-4}$, for all real values of x, $x \neq \dfrac{4}{3}$

(a) Find the range of f.

(b) Solve the equation $\mathrm{f}^{-1}(x) = 0$

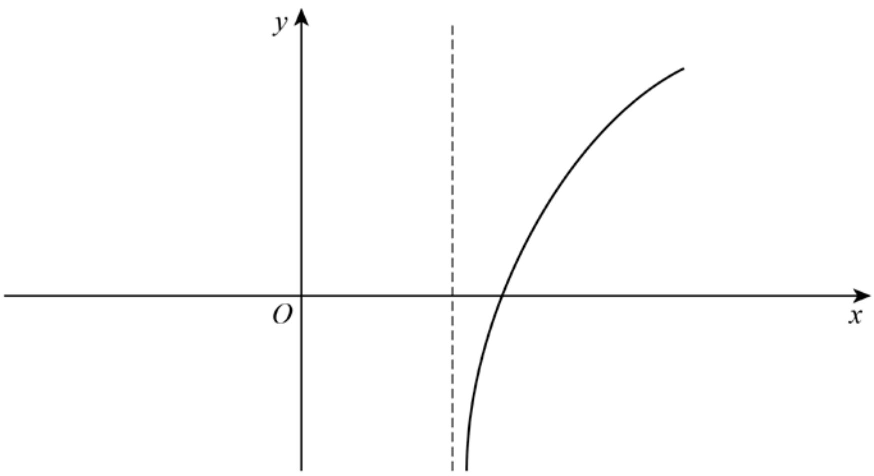
(c) (i) Find an expression for $\mathrm{gf}(x)$

(ii) Solve the equation $\mathrm{gf}(x) = 1$, giving your answer in exact form.

(iii) Find the exact value of x for which $\mathrm{gf}(x)$ is undefined.

2 The curve with equation $y = \mathrm{f}(x)$, where $\mathrm{f}(x) = \ln(2x-3)$, is sketched below.

The domain of f is $\{x \in \mathbb{R} : x > \frac{3}{2}\}$



(a) (i) Find $\mathrm{f}^{-1}(x)$

(ii) State the range of f^{-1}

(iii) Sketch the curve with equation $y = \mathrm{f}^{-1}(x)$, indicating the value of the y-intercept.

(b) The function g is defined by $\mathrm{g}(x) = \mathrm{e}^{2x} - 4$, for all real values of x

(i) Find $\mathrm{gf}(x)$, giving your answer in the form $(ax - b)^2 - c$, where a, b and c are integers.

(ii) Write down an expression for $\mathrm{fg}(x)$, and hence find the exact solution of the equation $\mathrm{fg}(x) = \ln 5$

B9	Understand the effect of simple transformations on the graph of $y = \mathrm{f}(x)$ including sketching associated graphs: $y = a\mathrm{f}(x)$, $y = \mathrm{f}(x) + a$, $y = \mathrm{f}(x + a)$ and $y = \mathrm{f}(ax)$ and combinations of these transformations.

Assessed at AS and A-level

Teaching guidance

Students should be able to:

- describe the following:

$y = a\,\mathrm{f}(x)$	Stretch in the y-direction scale factor a
$y = \mathrm{f}(x) + a$	Translation by the vector $\begin{pmatrix} 0 \\ a \end{pmatrix}$
$y = \mathrm{f}(x + a)$	Translation by the vector $\begin{pmatrix} -a \\ 0 \end{pmatrix}$
$y = \mathrm{f}(ax)$	Stretch in the x-direction scale factor $\frac{1}{a}$

- understand that applying different transformations may result in the same function.

Examples

1 Describe a single geometrical transformation that maps the graph of $y = 3^x$

(a) onto the graph of $y = 3^{2x}$

(b) onto the graph of $y = 3^{x+1}$

2 The graph of $y = x^2 - 6x + 9$ is translated by the vector $\begin{bmatrix} -3 \\ 0 \end{bmatrix}$

Find the equation of the translated graph.

3 The graph of $y = 2^x$ is mapped by a single transformation onto the graph of $y = 2^{x-2}$

(a) Fully describe the single transformation as a translation.

(b) Fully describe the single transformation as a stretch.

Only assessed at A-level

Teaching guidance

Students should be able to:

- apply two or more transformations to a function or describe a combination of two or more transformations that result in a given function.
- understand that applying transformations in a different order may result in two different functions.

Example

1 Describe a sequence of two geometrical transformations that maps the graph of $y = \cos^{-1} x$ onto the graph of $y = 2\cos^{-1}(x-1)$

B10

Decompose rational functions into partial fractions (denominators not more complicated than squared linear terms and with no more than three terms, numerators constant or linear).

Only assessed at A-level

Teaching guidance

Students should be able to:

- use the following forms:

$$\frac{px+q}{(ax+b)(cx+d)(ex+f)} \equiv \frac{A}{(ax+b)} + \frac{B}{(cx+d)} + \frac{C}{(ex+f)}$$

$$\frac{px+q}{(ax+b)(cx+d)^2} \equiv \frac{A}{(ax+b)} + \frac{B}{(cx+d)} + \frac{C}{(cx+d)^2}$$

- understand that the fractions may need to be simplified before partial fractions are found
- understand that partial fractions may be required for integration or in a binomial approximation.

Examples

1 (a) Express $\frac{2x+3}{4x^2-1}$ in the form $\frac{A}{2x-1} + \frac{B}{2x+1}$ where A and B are integers.

(b) Evaluate $\int_4^{12} \frac{2x+3}{4x^2-1}\,dx$ giving your answer in the form ln q, where q is a rational number.

2 (a) Express $\dfrac{1}{(3-2x)(1-x)^2}$ in the form $\dfrac{A}{3-2x}+\dfrac{B}{1-x}+\dfrac{C}{(1-x)^2}$

(b) Solve the differential equation

$$\frac{dy}{dx}=\frac{2\sqrt{y}}{(3-2x)(1-x)^2}$$

where $y = 0$ when $x = 0$, expressing your answer in the form

$$y^p = q\ln\left[f(x)\right]+\frac{x}{1-x}$$

where p and q are constants.

3 $$f(x)=\frac{7x-1}{(1+3x)(3-x)}$$

(a) Express $f(x)$ in the form $\dfrac{A}{3-x}+\dfrac{B}{1+3x}$ where A and B are integers.

(b) (i) Find the first three terms of the binomial expansion of $f(x)$ in the form $a + bx + cx^2$, where a, b and c are rational numbers.

(ii) State why the binomial expansion cannot be expected to give a good approximation of $f(0.4)$

AQA

B11	Use of functions in modelling, including consideration of limitations and refinements of the models.

Only assessed at A-level

Teaching guidance

Students should be able to:

- suggest how a model could be improved – this is a refinement
- give suggestions as to when a particular model might break down or why it is only appropriate over a particular range of values – these are limitations.

Notes

- Functions may be used in the formulation of a differential equation or arise from the solution of a differential equation.
- Modelling questions are an ideal opportunity to use calculators. Typically, exact answers will be inappropriate and thus equations can be solved using the equation solving facility of a standard scientific calculator.
- Considering the long-term behaviour of a function will often be a useful technique.

Examples

1 The total number of views, V, of a viral video clip that is released on the internet can be modelled by the formula

$$V = 150 \times 2^{d}$$

where d is the number of days after the video clip has been released.

(a) How many days does it take for the video to reach 1 million views?

(b) Explain why this model will eventually be inappropriate.

2 A zoologist is studying a population of 100 rodents introduced to a small island.

In order to model the size of the population, she assumes that the rate of increase of the number of rodents, $\frac{dN}{dt}$, at time t, will be proportional to the size of the population, N

This leads the zoologist to the model $N = Ae^{kt}$

(a) State the value of A

(b) Describe what happens to the population of rodents, modelled in this way, as time increases.

(c) State one criticism of this model and explain how the model could be improved.

C Coordinate geometry in the (x, y) plane

C1

Understand and use the equation of a straight line, including the forms $y - y_1 = m(x - x_1)$ and $ax + by + c = 0$; gradient conditions for two straight lines to be parallel or perpendicular.

Be able to use straight line models in a variety of contexts.

Assessed at AS and A-level

Teaching guidance

Students should:

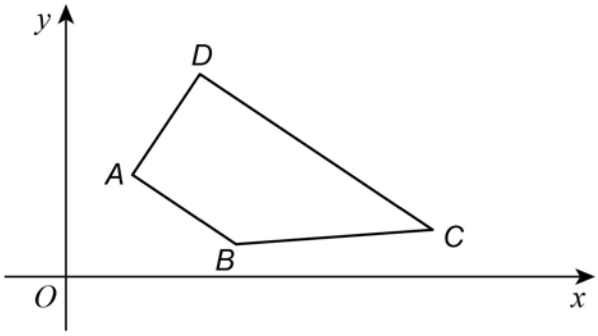
- be able to solve problems using gradients, midpoints and the distance between two points, including the form $y = mx + c$ and the forms $y = a$ and $x = b$ for horizontal and vertical lines
- know that the product of the gradients of two perpendicular lines is -1
- Understand necessary and sufficient conditions for a quadrilateral to be a square, rectangle, rhombus, parallelogram, kite or trapezium and be able to apply understanding of straight lines to these.

Notes

- In questions where the equation of a line is to be found, any correct form will be acceptable, unless specified in the question.
- However, trivial simplifications left undone in equations are likely to be penalised, eg $y - -2 = \frac{2}{4}(x-1)$ should be simplified to $y + 2 = \frac{1}{2}(x-1)$

Examples

1 The trapezium $ABCD$ is shown below.



The line AB has equation $2x + 3y = 14$ and DC is parallel to AB.

(a) The point D has coordinates (3, 7)

(i) Find an equation of the line DC.

(ii) The angle BAD is a right angle.

Find an equation of the line AD, giving your answer in the form $mx + ny + p = 0$, where m, n and p are integers.

(b) The line BC has equation $5y - x = 6$. Find the coordinates of B.

2 The point A has coordinates (–1, 2) and the point B has coordinates (3, –5)

(a) Find an equation of the line AB, giving your answer in the form $px + qy = r$, where p, q and r are integers.

(b) The midpoint of AB is M. Find an equation of the line which passes through M and which is perpendicular to AB.

(c) The point C has coordinates $(k, 2k + 3)$

Given that the distance from A to C is $\sqrt{13}$ find the possible values of the constant k.

C2

Understand and use the coordinate geometry of the circle including using the equation of a circle in the form $(x-a)^2 + (y-b)^2 = r^2$; completing the square to find the centre and radius of a circle; use the following properties:

- **the angle in a semicircle is a right angle**
- **the perpendicular from the centre to a chord bisects the chord**
- **the radius of a circle at a given point on its circumference is perpendicular to the tangent to the circle at that point.**

Assessed at AS and A-level

Teaching guidance

Students should be able to:

- find the equation of a tangent or normal at a point
- find relevant gradients using the coordinates of appropriate points.

Note: implicit differentiation of the equation of a circle will not be required at AS, but could be expected at A-level.

Examples

1 A circle with centre C has equation $x^2 + y^2 - 10x + 12y + 41 = 0$

The point A (3, –2) lies on the circle.

(a) (i) Find the coordinates of C.

(ii) Show that the circle has radius $n\sqrt{5}$, where n is an integer.

(b) Find the equation of the tangent to the circle at point A, giving your answer in the form $x + py = q$, where p and q are integers.

(c) The point B lies on the tangent to the circle at A and the length of BC is 6.

Find the length of AB.

2 The points P (4, 3), Q (6, 7) and R (12, 4) lie on a circle, C.

(a) Show that PQ and QR are perpendicular.

(b) Find the length of PR, giving your answer as a surd.

(c) Find the equation of the circle C.

3 A circle has equation $x^2 + y^2 - 4x - 14 = 0$

(a) (i) Find the coordinates of the centre of the circle.

(ii) Find the radius of the circle in the form $p\sqrt{2}$, where p is an integer.

(b) A chord of the circle has length 8. Find the perpendicular distance from the centre of the circle to this chord.

(c) A line has equation $y = 2k - x$ where k is a constant.

(i) Show that the x-coordinate of any point of intersection of the line and the circle satisfies the equation

$$x^2 - 2(k + 1)x + 2k^2 - 7 = 0$$

(ii) Find the values of k for which the equation

$$x^2 - 2(k + 1)x + 2k^2 - 7 = 0$$

has equal roots.

(iii) Describe the geometrical relationship between the line and the circle when k takes either value found in part (c)(ii).

C3 Understand and use the parametric equations of curves and conversion between Cartesian and parametric forms.

Only assessed at A-level

Teaching guidance

Students should:

- understand that it is not always expected that the resulting Cartesian equation will be in explicit form

  Note: any correct form will be acceptable, unless stated in the question
- understand that parametric equations using trigonometric terms, often require the use of appropriate identities
- be able to answer questions requiring the gradient of a parametric curve or the tangent or normal to such a curve
- be able to find the area enclosed by a curve given in parametric form and the x-axis.

Examples

1 A curve is defined by the parametric equations

$$x = 3 - 4t \qquad y = 1 + \frac{2}{t}$$

Show that the Cartesian equation of the curve can be written as

$$(x - 3)(y - 1) + 8 = 0$$

2 A curve is given by the parametric equations

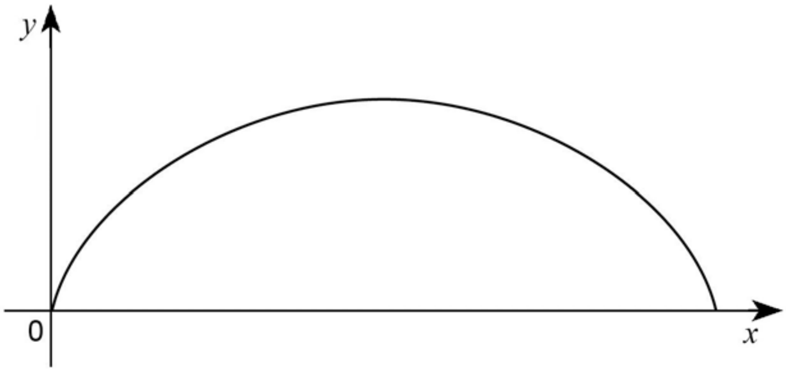
$$x = \cos\theta \qquad y = \sin 2\theta$$

Show that the Cartesian equation of the curve can be written as

$$y^2 = kx^2(1 - x^2)$$

where k is an integer.

3 The curve with parametric equations $x = 1 + \cos\theta$ and $y = \sin\theta$, $0 \le \theta \le \pi$ is shown below.



Find a Cartesian equation of the curve in the form $x^n + y^m = kx$

C4

Use parametric equations in modelling in a variety of contexts.

Only assessed at A-level

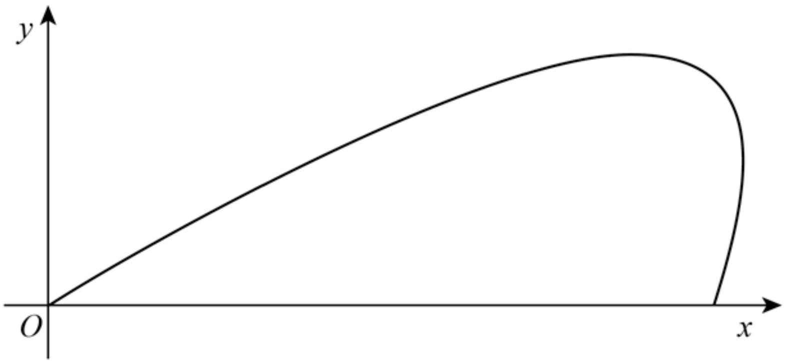
Teaching guidance

Students should be able to use parametric equations to describe the motion of a particle in the (x, y) plane, for example $x = 4t$, $y = 3t - 4.9t^2$, using the acceleration due to gravity as 9.8 m s^{-2} for a particle subject only to the force of its own weight, projected from the origin at time $t = 0$ with velocity $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$

Examples

1 A ball is thrown from an origin in a strong wind so that its horizontal and vertical position at time t is given by $x = 25t - 10t^2$ and $y = 15t - 10t^2$

The trajectory of the ball is shown in the diagram.



(a) Find the time it takes for the ball to hit the ground.

(b) Find $\dfrac{dy}{dx}$ in terms of t

(c) Use your answers to parts (a) and (b) to find the angle at which the ball hits the ground.

D Sequences and series

D1

Understand and use the binomial expansion of $(a + bx)^n$ for positive integer n; the notations $n!$ and nCr; link to binomial probabilities.

Extend to any rational n, including its use for approximation; be aware that the expansion is valid for $\left|\frac{bx}{a}\right| < 1$ (Proof not required.)

Assessed at AS and A-level

Teaching guidance

Students should be able to:

- answer questions requiring the full expansion of expressions of the form $(a+bx)^n$, where n is a small positive integer
- find the coefficients of particular powers of x (complete expansion not required)
- understand factorial notation.

Notes

- The notations $\binom{n}{r}$, ${}_nC_r$ and ${}^nC_r$ must all be recognised. Any of these may be used.
- The x in $(a+bx)^n$ may be a simple function of x, eg $\left(2-\frac{1}{x}\right)^4$

Examples

1 Show that the expansion of

$$(1 + 3x)^4 - (1 + 4x)^3$$

can be written in the form

$$px^2 + qx^3 + rx^4$$

where p, q and r are integers.

2 Find the exact value of $\int_1^2 \left[\left(2+x^{-2}\right)^3 + (2-x^{-2})^3\right] \mathrm{d}x$

Only assessed at A-level

Teaching guidance

Students should be familiar with and be able to use the formulae:

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{r}a^{n-r}b^r + \ldots + b^n \qquad \left(n \in \mathbb{Z}^+\right)$$

where: $\binom{n}{r} = {}^{n}\mathrm{C}_r = \dfrac{n!}{r!(n-r)!}$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1.2}x^2 + \ldots + \frac{n(n-1)\ldots(n-r+1)}{1.2\ldots r}x^r + \ldots \qquad \left(|x| < 1, n \in \mathbb{Q}\right)$$

Note: the binomial expansion of $(1 + x)^n$ is valid for all real n, but it is only required for rational n in this specification.

Examples

1 (a) Find the binomial expansion of $\dfrac{1}{1+3x}$ up to the term in x^3

(b) Express $\dfrac{1+4x}{(1+x)(1+3x)}$ in partial fractions.

(c) (i) Find the binomial expansion of $\dfrac{1+4x}{(1+x)(1+3x)}$ up to the term in x^3

(ii) Find the range of values of x for which the binomial expansion of $\dfrac{1+4x}{(1+x)(1+3x)}$ is valid.

2 (a) Find the binomial expansion of $(8+6x)^{\frac{2}{3}}$ up to and including the term in x^2

(b) Use your answer from part (a) to find an estimate for $\sqrt[3]{100}$ in the form $\dfrac{a}{b}$, where a and b are integers.

Fully justify your answer.

D2

Work with sequences including those given by a formula for the n^{th} term and those generated by a simple relation of the form $x_{n+1} = \mathrm{f}(x_n)$; increasing sequences; decreasing sequences; periodic sequences.

Only assessed at A-level

Teaching guidance

Students should:

- understand and be able to use notation such as u_n
- understand that an increasing sequence will be one where $u_{n+1} > u_n$ for all n

  eg

  $u_{n+1} = 2u_n, u_1 = 5$

  $u_n = \dfrac{n}{n+1}$
- understand that a decreasing sequence will be one where $u_{n+1} < u_n$, for all n

  eg

  $u_{n+1} = 0.5u_n,\ u_1 = 5$

  $u_n = 2^{-n}$
- know that a periodic sequence repeats over a fixed interval, ie $u_{n+a} = u_n$, for all n, for a constant a and that a is the period of the sequence

  eg

  1, 2, 3, 4, 1, 2, 3, 4, …

  $u_n = \sin\left(\dfrac{\pi n}{2}\right)$
- be able to find the limit, L, of a sequence $u_{n+1} = \mathrm{f}(u_n)$ as $n \to \infty$ by setting $L = \mathrm{f}(L)$

Examples

1 A sequence is defined by

$$u_{n+1} = pu_n + q$$

where p and q are constants.

The first three terms of the sequence are given by:

$u_1 = 200$ $u_2 = 150$ $u_3 = 120$

(a) Show that p = 0.6 and find the value of q.

(b) Find the value of u_4

(c) The limit of u_n as n tends to infinity is L.

Write down an equation satisfied by L and hence find the value of L.

2 The nth term of a sequence is defined by

$$u_n = \frac{n}{n+1}$$

Prove that u_n is an increasing sequence.

3 When the fraction $\frac{1250}{999}$ is written as a decimal its digits form a periodic sequence.

What digit is in the 1000th decimal place?

D3

Understand and use sigma notation for sums of series.

Only assessed at A-level

Teaching guidance

Students should be able to:

- use $\sum$ to indicate the sum of a series, eg

  $$\sum_{r=1}^{5}(2r+1)=3+5+7+9+11$$

  $$u_n=2^{-n}, \sum_{n=0}^{\infty}u_n=1+\frac{1}{2}+\frac{1}{4}+\ldots=2$$

- understand and use the notation S_n for the sum of a series
- use a calculator to find the sum of a series.

Examples

1 Find the value of

$$\sum_{r=1}^{4}\ln(2^r)$$

giving your answer in the form $p\ln 2$

2 Find the exact value of

$$\sum_{r=1}^{\infty}\frac{\sqrt{2}}{2^r}$$

3 Which of these has a different value from the other three?

Circle your answer.

$\sum_{r=1}^{k}(3r+4)$ $\quad$ $\sum_{r=2}^{k+1}(3r+1)$ $\quad$ $\sum_{r=1}^{k-1}(3r+7)$ $\quad$ $\sum_{r=5}^{k+4}(3r-8)$

D4

Understand and work with arithmetic sequences and series, including the formulae for nth term and the sum to n terms.

Only assessed at A-level

Teaching guidance

Students should know the formula for the nth term of an arithmetic sequence and be able to use the formulae for the sum of the first n terms of an arithmetic series

$u_n = a + (n-1)d$	nth term of the sequence
$S_n = \frac{n}{2}(a+l)$	Sum of first n terms using first and last term
$S_n = \frac{n}{2}(2a+(n-1)d)$	Sum of first n terms using first term and common difference

where a is the first term, d is the common difference and l is the last term of the sequence.

Notes:

Technically a series is the (infinite) sum of the terms of a sequence:

Sequence	8, 15, 22, 29, 36, …
Series	8 + 15 + 22 + 29 + 36 + …

Questions can refer to the sum of the first n terms of an arithmetic sequence or to the sum of the first n terms of an arithmetic series; both have the same meaning.

Examples

1 The first term of an arithmetic series is 1. The common difference of the series is 6

(a) Find the 10th term of the series.

(b) The sum of the first n terms of the series is 7400

(i) Show that $3n^2 - 2n - 7400 = 0$

(ii) Find the value of n

2 The 25th term of an arithmetic series is 38

The sum of the first 40 terms of the series is 1250

(a) Show that the common difference of this series is 1.5

(b) Find the number of terms in the series which are less than 100

3 This is an arithmetic series

51 + 58 + 65 + 72 + … + 1444

(a) Find the 101st term of the series.

(b) Find the sum of the last 100 terms of the series.

D5

Understand and work with geometric sequences and series including the formulae for the nth term and the sum of a finite geometric series; the sum to infinity of a convergent geometric series, including the use of $|r| < 1$; modulus notation.

Only assessed at A-level

Teaching guidance

Students should:

- know the formula for the nth term and be able to use the sum formulae:

$u_n = ar^{n-1}$	nth term of the sequence
$S_n = \dfrac{a(1-r^n)}{1-r}\left(= \dfrac{a(r^n-1)}{r-1}\right)$	sum of first n terms
$S_\infty = \dfrac{a}{1-r}, \|r\| < 1$	sum to infinity

where a is the first term and r is the common ratio

- understand the condition for convergence of a geometric series
- understand the term 'convergent'.

Examples

1 A geometric series has first term 80 and common ratio $\frac{1}{2}$

(a) Find the sum of the first 12 terms of the series, giving your answer to two decimal places.

(b) Find the sum to infinity of the series.

2 The first three terms of a geometric sequence are x, $x + 6$ and $x + 9$

Find the common ratio of this sequence, giving your answer as a fraction in its simplest form.

3 An infinite geometric series has common ratio r. The sum to infinity of the series is five times the first term of the series.

(a) Show that $r = 0.8$

(b) Given that the first term of the series is 20, find the least value of n such that the nth term of the series is less than 1

D6 Use sequences and series in modelling.

Only assessed at A-level

Teaching guidance

Students should be able to answer questions set within a context, eg compound interest.

Examples

1 A ball is dropped from a height of 1 metre above the ground.

Each time it hits the ground it bounces to a height of $\frac{3}{4}$ the distance it fell before the bounce.

(a) Show that the distance travelled by the ball between the first and second bounce is 1.5 metres.

(b) Find the total distance travelled by the ball after it is dropped.

2 A maths graduate begins a job with a starting salary of £26 000. She has been promised an annual pay rise of 5% of the previous year's salary.

(a) How much should she expect to earn in her third year?

(b) If she stays in the same job for seven years, how much would she earn in total over this time?

E Trigonometry

E1

Understand and use the definitions of sine, cosine and tangent for all arguments; the sine and cosine rules; the area of a triangle in the form $\frac{1}{2}ab\sin C$

Work with radian measure, including use for arc length and area of sector.

Assessed at AS and A-level

Teaching guidance

Students should:

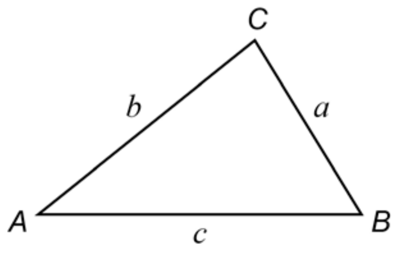
- know and be able to apply the following rules:

  In any triangle *ABC*

  • area of triangle $\frac{1}{2}ab\sin C$

  • sine rule $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

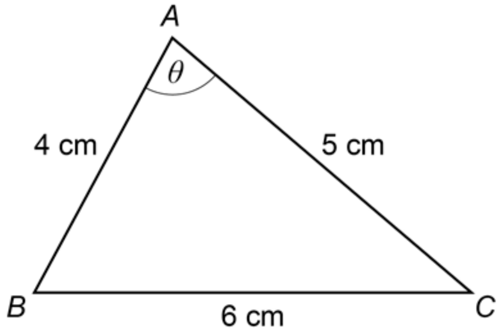
  • cosine rule $a^2 = b^2 + c^2 - 2bc\cos A$



- be aware of the ambiguous case that can arise from the use of the sine rule.

Examples

1 The triangle *ABC*, shown in the diagram, is such that *BC* = 6 cm, *AC* = 5 cm and *AB* = 4 cm. The angle *BAC* is θ.



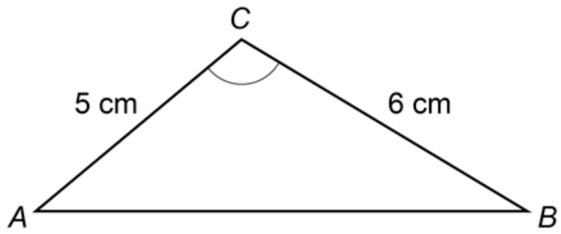
(a) Show that $\cos\theta = \frac{1}{8}$

(b) Find the exact value of the area of the triangle *ABC*.

2 Angle θ is such that $\sin\theta = \frac{1}{7}$ and $0 < \theta < 90°$

Find the exact value of $\tan\theta$

3 The diagram shows a triangle *ABC*.



The lengths of the sides *AC* and *BC* are 5 cm and 6 cm respectively.

The area of triangle *ABC* is 12.5 cm^2, and angle *ACB* is obtuse.

Find the length of *AB*, giving your answer to two significant figures.

Only assessed at A-level

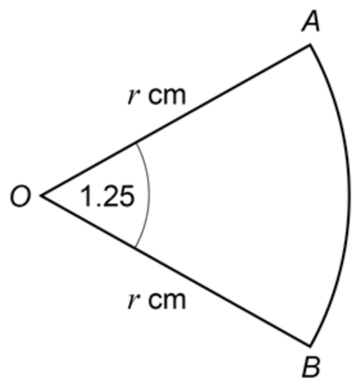
Teaching guidance

Students should:

- understand and be able to use radian measure.
- know that 2π radians $= 360°$
- know and be able to use $l = r\theta,\ A = \frac{1}{2}r^2\theta$

Examples

1 The diagram shows a sector of *OAB* of a circle with centre *O* and radius *r* cm.



The angle *AOB* is 1.25 radians. The perimeter of the sector is 39 cm.

(a) Show that $r = 12$

(b) Calculate the area of the sector *OAB*.

E2

Understand and use the standard small angle approximations of sine, cosine and tangent $\sin\theta \approx \theta$, $\cos\theta \approx 1 - \frac{\theta^2}{2}$ and $\tan\theta \approx \theta$ where θ is in radians.

Only assessed at A-level

Teaching guidance

Students should be able to:

- understand and use the standard small angle approximations of sine, cosine and tangent when differentiating sine or cosine from first principles
- use standard small angle approximations to deduce approximations for other functions.

Notes

- Students should be aware that a reasonable validity for the small angle approximations is for angles less than 0.5 radians. The range of validity can be explored in greater depth, but does not need to be known for this specification.

Examples

1 (a) Show, using a small angle approximation, that $\sec x \approx \dfrac{2}{2-x^2}$

(b) Hence, find the first two terms of the binomial expansion for $\sec x$

(c) Using your binomial expansion find an approximate value for $\sec 0.1$, giving your answer to 5 decimal places.

2 (a) Using a compound angle identity write down an expression for $\sin(x+h)$

(b) Using small angle approximations for $\sin(h)$ and $\cos(h)$, and your answer to part (a), find

$$\lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h}$$

E3	**Understand and use the sine, cosine and tangent functions; their graphs, symmetries and periodicity.** Know and use exact values of sin and cos for $0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}, \pi$ and multiples thereof, and exact values of tan for $0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \pi$ and multiples thereof.

Assessed at AS and A-level

Teaching guidance

Students should be able to:

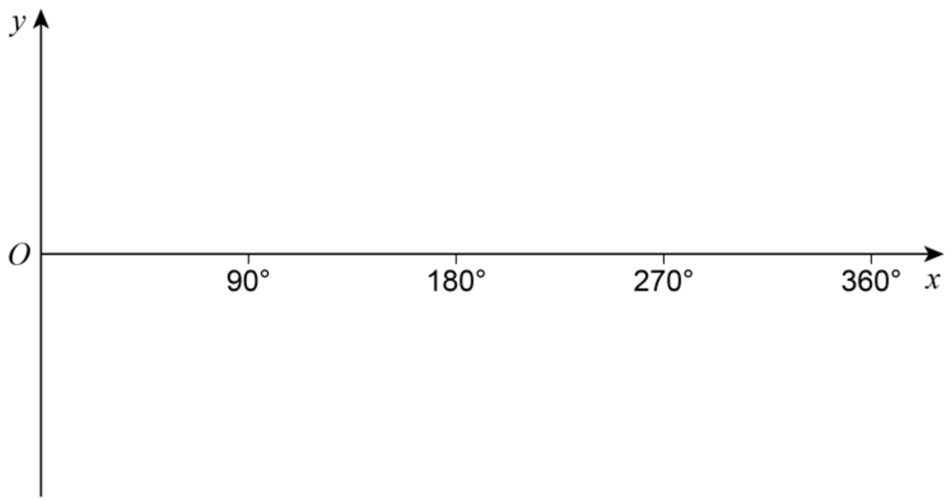
- understand and use vertical asymptotes of a tangent graph
- carry out simple transformations (as given in section B9) of the graphs of the sine, cosine and tangent functions.

At A-level combinations of transformations may be used.

Note: radians will not be required at AS.

Examples

1 (a) On the axes given below, sketch the graph of $y = \tan x$, for $0° \le x \le 360°$



(b) Solve the equation $\tan x = -1$, giving all the values of x in the interval $0° \le x \le 360°$

2 Find the two solutions of the equation $\sin x = \sin\sqrt{41}$ for $0° \leq x \leq 360°$

Give your answers in exact form.

Teaching guidance

Only assessed at A-level

Students should:

- understand and be able to use radians
- understand the phrase ‘exact value’ and use their calculators appropriately when an exact value is required.

Examples

1 Given that $\sin a = \dfrac{1}{3}$ and $0 < a < \dfrac{\pi}{2}$, find the exact value of $\sin\left(a + \dfrac{\pi}{6}\right)$

2 (a) Sketch the graph of $y = \cos x$ in the interval $0 \leq x \leq 2\pi$

(b) State the values of the intercepts with the coordinate axes.

E4

Understand and use the definitions of secant, cosecant and cotangent and of arcsin, arccos and arctan; their relationships to sine, cosine and tangent; understanding their graphs; their ranges and domains.

Only assessed at A-level

Teaching guidance

Students should:

- know and be able to use the following functions and their graphs:

Secant	Cosecant	Cotangent
$\sec x \equiv \dfrac{1}{\cos x}$	$\operatorname{cosec} x \equiv \dfrac{1}{\sin x}$	$\cot x \equiv \dfrac{1}{\tan x} \equiv \dfrac{\cos x}{\sin x}$

- be aware of both notations for inverse functions, eg $\arcsin x$ or $\sin^{-1} x$;
- know the domains and ranges of these functions, in particular

$$-\frac{\pi}{2} \le \sin^{-1} x \le \frac{\pi}{2}$$

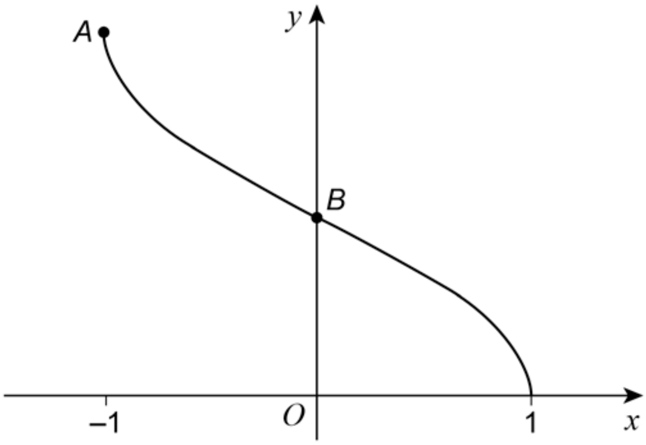
$$0 \le \cos^{-1} x \le \pi$$

$$-\frac{\pi}{2} < \tan^{-1} x < \frac{\pi}{2}$$

- understand how to sketch the graphs of the inverse trig functions by reflecting relevant sections of the trigonometric graphs in the line $y = x$
- be able to apply simple transformations (as defined in section B9) and combinations of these to graphs of all these functions.

Examples

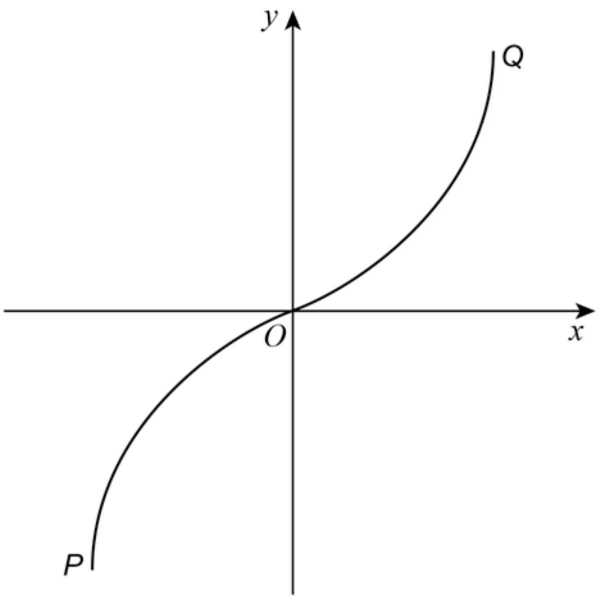
1 The diagram shows the curve $y = \cos^{-1} x$ for $-1 \le x \le 1$



Write down the exact coordinates of points *A* and *B*.

2 Sketch the curve with equation $y = \operatorname{cosec} x$ for $0 < x < \pi$

3 (a) The sketch shows the graph of $y = \sin^{-1} x$

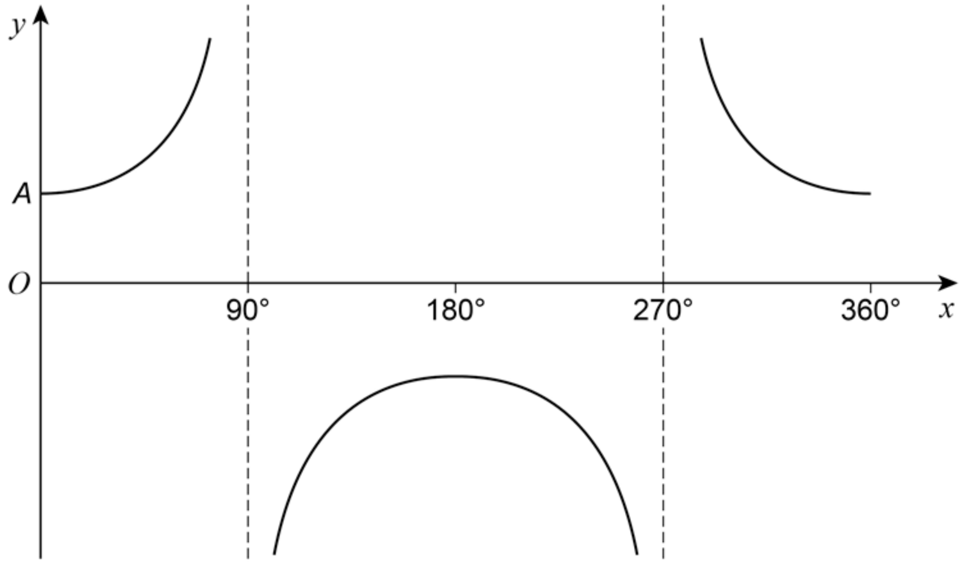


Write down the coordinates of the points *P* and *Q*, the end points of the graph.

(b) Sketch the graph of $y = \sin^{-1}(x - 1)$

4 Solve the equation sec $x = 5$, giving all the values of x in the interval $0 \le x \le 2\pi$ to two decimal places.

5 (a) The diagram shows the graph of $y = \sec x$ for $0° \leq x \leq 360°$



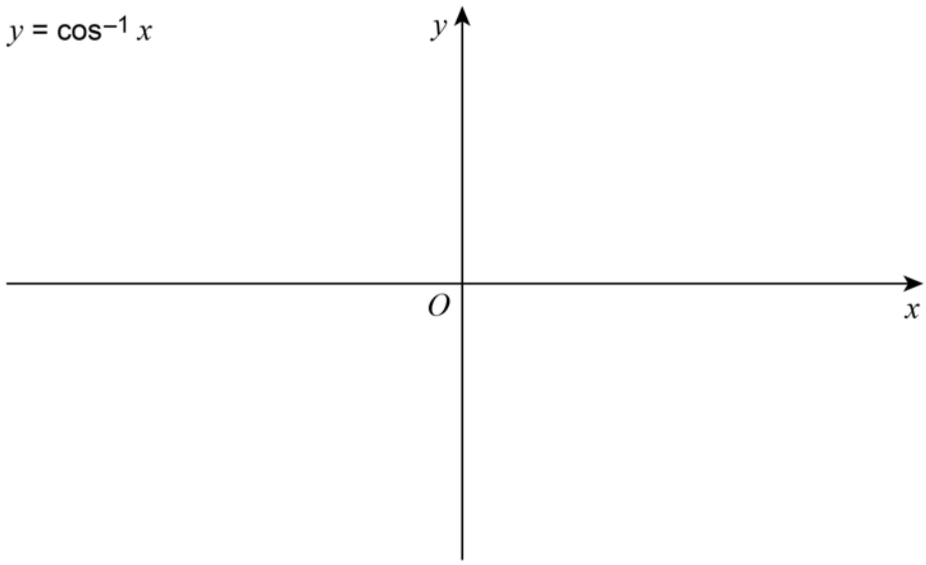
(i) The curve meets the y-axis at A. State the y-coordinate of A.

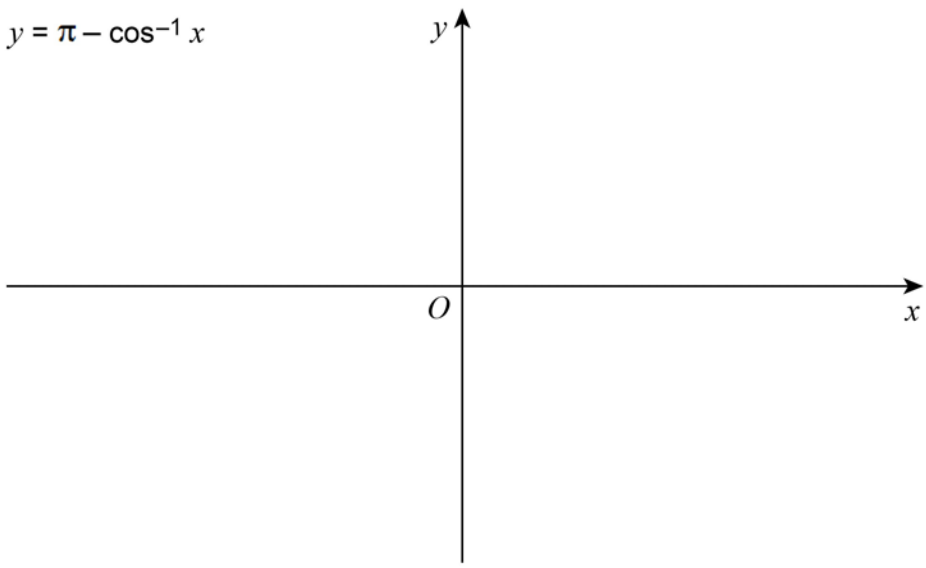
(ii) Sketch the graph of $y = \sec(2x) + 2$ for $0° \leq x \leq 360°$

(b) Solve the equation $\sec x = 2$ giving all the values of x in the interval $0° \leq x \leq 360°$

6 (a) Sketch the graph of $y = \cos^{-1} x$, where y is in radians on the first set of axes below. State the coordinates of the end points of the graph.

(b) Sketch the graph of $y = \pi - \cos^{-1} x$, where y is in radians on the second axes below. State the coordinates of the end points of the graph.





AQA

E5

Understand and use $\tan\theta \equiv \dfrac{\sin\theta}{\cos\theta}$.

Understand and use $\sin^2\theta + \cos^2\theta \equiv 1$; $\sec^2\theta \equiv 1 + \tan^2\theta$ and $\operatorname{cosec}^2\theta \equiv 1 + \cot^2\theta$

Assessed at AS and A-level

Teaching guidance

Students should be able to:

- use $\tan\theta \equiv \dfrac{\sin\theta}{\cos\theta}$ to solve equations or find exact values
- use $\sin^2\theta + \cos^2\theta \equiv 1$ to solve equations or find exact values.

Examples

1 (a) Given that $6\tan\theta\sin\theta = 5$, show that $6\cos^2\theta + 5\cos\theta - 6 = 0$

(b) Hence solve the equation $6\tan(3x)\sin(3x) = 5$, giving all values of x to the nearest degree in the interval $0° \le x \le 180°$

2 $(\tan\theta + 1)(\tan^2\theta - 3) = 0$

(a) Find the possible values of $\tan\theta$

(b) Hence solve the equation $(\tan\theta + 1)(\sin^2\theta - 3\cos^2\theta) = 0$, giving all solutions for θ, in degrees, in the interval $0° \le \theta \le 180°$

3 Given that $\sin\theta = \dfrac{2}{7}$ and θ is obtuse, find:

(a) the exact value of $\cos\theta$

(b) the exact value of $\tan\theta$

Only assessed at A-level

Teaching guidance

Students should be able to:

- use $\sec^2\theta \equiv 1+\tan^2\theta$ and $\operatorname{cosec}^2\theta \equiv 1+\cot^2\theta$ to solve equations or find exact values
- use identities to perform integration, eg $\int \tan^2 x \, \mathrm{d}x$
- prove identities.

Examples

1 (a) Solve

$$\tan^2\theta = 3(3-\sec\theta)$$

giving all solutions to the nearest 0.1º in the interval $0° < \theta < 360°$

(b) Hence solve

$$\tan^2(4x-10°) = 3\left[3-\sec(4x-10°)\right]$$

giving all solutions to the nearest 0.1º in the interval $0° < x < 90°$

AQA

E6	Understand and use double angle formulae; use of formulae for $\sin(A \pm B)$, $\cos(A \pm B)$ and $\tan(A \pm B)$; understand geometrical proofs of these formulae. Understand and use expressions for $a\cos\theta + b\sin\theta$ in the equivalent forms of $r\cos(\theta \pm \alpha)$ or $r\sin(\theta \pm \alpha)$

Only assessed at A-level

Teaching guidance

Students should be able to:

- understand and use the addition formulae:

  $\sin(A \pm B) = \sin(A)\cos(B) \pm \cos(A)\sin(B)$

  $\cos(A \pm B) = \cos(A)\cos(B) \mp \sin(A)\sin(B)$

  $\tan(A \pm B) = \dfrac{\tan(A) \pm \tan(B)}{1 \mp \tan(A)\tan(B)}$

- understand how these formulae can be used to derive the double angle formulae:

  $\sin(2A) = 2\sin(A)\cos(A)$

  $$\begin{aligned}\cos(2A) &= \cos^2(A) - \sin^2(A)\\ &= 2\cos^2(A) - 1\\ &= 1 - 2\sin^2(A)\end{aligned}$$

- use double angle formulae to solve equations and perform integration
- use the harmonic forms $r\cos(\theta \pm \alpha)$ or $r\sin(\theta \pm \alpha)$ to solve equations or describe features of the resulting wave function, eg maximum or minimum, amplitude, etc.

Notes

- Whilst the correct formal notation is $\equiv$, it is common practice to use = when the nature of the identity is understood. Thus either $\equiv$ or = will be accepted in exams.

Examples

1 Show that $\int_0^{\frac{\pi}{2}} \cos^2 x \, dx = \frac{\pi}{a}$, where a is an integer.

2 $3\cos\theta - 2\sin\theta \equiv R\cos(\theta + \alpha)$, where $R > 0$ and $0° < \alpha < 90°$

(a) Find the value of R.

(b) Show that $\alpha \approx 33.7°$

(c) Hence write down the maximum value of $3\cos\theta - 2\sin\theta$ and find a positive value of θ at which this maximum value occurs.

3 The polynomial $\mathrm{f}(x)$ is defined by $\mathrm{f}(x) = 4x^3 - 11x - 3$

(a) Use the factor theorem to show that $(2x + 3)$ is a factor of $\mathrm{f}(x)$

(b) Write $\mathrm{f}(x)$ in the form $(2x + 3)(ax^2 + bx + c)$

(c) Hence find all solutions of the equation $2\cos 2\theta \sin\theta + 9\sin\theta + 3 = 0$ in the interval $0° < \theta < 360°$, giving your solutions to the nearest degree.

4 (a) Show that $\cot x - \sin 2x \equiv \cot x \cos 2x$ for $0° < x < 180°$

(b) Hence, or otherwise, solve the equation $\cot x - \sin 2x = 0$ in the interval $0° < x < 180°$

5 Angle α is acute and $\cos\alpha = \dfrac{3}{5}$ Angle β is obtuse and $\sin\beta = \dfrac{1}{2}$

Show that

$$\tan(\alpha + \beta) = \frac{m\sqrt{3} - n}{n\sqrt{3} + m}$$

where m and n are integers.

E7

Solve simple trigonometric equations in a given interval, including quadratic equations in sin, cos and tan and equations involving multiples of the unknown angle.

Assessed at AS and A-level

Teaching guidance

Students should be able to:

- understand and solve simple trigonometric equations
- answer questions that require them to give solutions in degrees.

Examples

1 Solve the equation $\sin(\theta - 30°) = 0.7$, giving your answers to the nearest 0.1º in the interval $0° \le \theta \le 360°$

2 Write down all solutions of the equation tan x = tan 61º in the interval $0° \le x \le 360°$

3 Solve the equation sin 2x = sin 48º, giving the values of x in the interval $0° \le x \le 360°$

4 (a) Given that

$$\frac{\cos^2 x + 4\sin^2 x}{1 - \sin^2 x} = 7$$

show that

$$\tan^2 x = \frac{3}{2}$$

(b) Hence solve

$$\frac{\cos^2 2\theta + 4\sin^2 2\theta}{1 - \sin^2 2\theta} = 7$$

in the interval

$$0° \le \theta \le 180°$$

giving your values of θ to the nearest degree.

Only assessed at A-level

Teaching guidance

Students should be able to:

- understand and solve trigonometric equations using formulae from section E6, eg $\sec\left(2x+\frac{\pi}{3}\right)=\sqrt{2}$
- answer questions in radians.

Examples

1 Solve $\tan\frac{1}{2}x=3$ in the interval $0<x<4\pi$, giving your answers to three significant figures.

2 Solve the equation $\cos\theta(\sin\theta-3\cos\theta)=0$ in the interval $0<\theta<2\pi$, giving your answers to three significant figures.

3 (a) Show that the equation $2\operatorname{cosec}^2 x=5-5\cot x$ can be written in the form

$$2\cot^2 x+5\cot x-3=0$$

(b) Hence show that $\tan x = 2$ or $\tan x=-\frac{1}{3}$

(c) Hence, or otherwise, solve the equation $2\operatorname{cosec}^2 x=5-5\cot x$, giving all values of x to one decimal place in the interval $-\pi\le x\le\pi$

4 (a) Solve the equation $\sec x=5$, giving all the values of x in the interval $0\le x\le 2\pi$ to two decimal places.

(b) Solve the equation $\tan^2 x=3\sec x+9$, giving all the values of x in the interval $0\le x\le 2\pi$ in radians to two decimal places.

E8	Construct proofs involving trigonometric functions and identities.

Only assessed at A-level

Teaching guidance

Students should understand and be able to use the following:

$\sin^2\theta + \cos^2\theta \equiv 1$ $\quad\quad$ $\sec^2\theta \equiv 1 + \tan^2\theta$ $\quad\quad$ $\operatorname{cosec}^2\theta \equiv 1 + \cot^2\theta$

Notes

- Unless stated in the question, the above may be quoted without proof.
- Whilst the proof of an identity might typically start with the left-hand side (LHS) and deduce the right (RHS) it is equally acceptable to work from the RHS to the LHS.
- Proofs that start from both sides and meet in the middle can also be successful, but the two chains of reasoning must clearly link together.
- Proofs that transform the original identity into another, perhaps by multiplying through by a denominator, can also be successful, but often are not because of a lack of rigour.

Examples

1 Prove the identity $(\tan x + \cot x)^2 \equiv \sec^2 x + \operatorname{cosec}^2 x$

2 Prove the identity $\dfrac{1-\sin x}{\cos x} + \dfrac{\cos x}{1-\sin x} \equiv 2\sec x$

3 Prove the identity $\dfrac{\sec^2 x}{(\sec x + 1)(\sec x - 1)} \equiv \operatorname{cosec}^2 x$

E9

Use trigonometric functions to solve problems in context, including problems involving vectors, kinematics and forces.

Only assessed at A-level

Teaching guidance

Students should be able to:

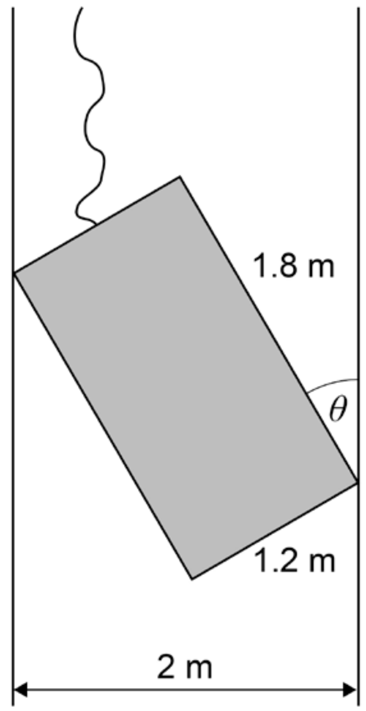
- solve problems using any of the techniques from sections E1 to E8 on their own or in combination
- select for themselves the appropriate technique for solving a problem.

Examples

1 A crane is lowering a heavy crate down a mine shaft when the crate scrapes the side of the mine shaft, twists and becomes stuck.

The mine shaft has a width of 2 metres and the crate is 1.8 metres tall by 1.2 metres wide.

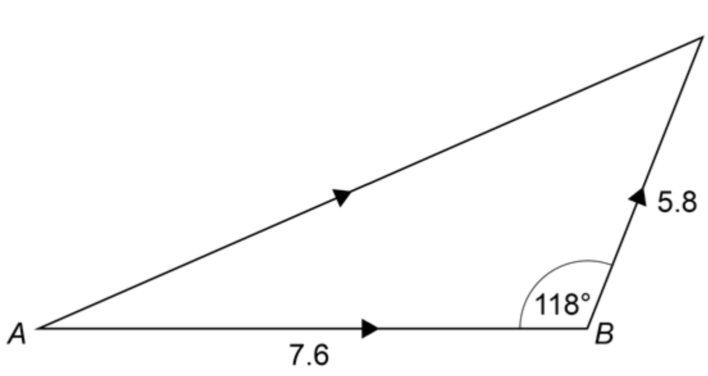
The angle between the wall of the mine shaft and the side of the crate is θ, as shown in the diagram.



(a) Show that $9\sin\theta + 6\cos\theta = 10$

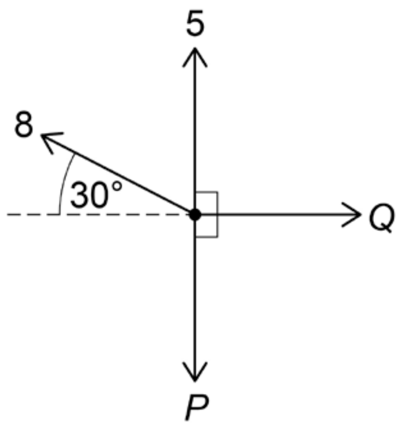
(b) Hence find the value of θ

2 Two vectors $\overrightarrow{AB}$ and $\overrightarrow{BC}$ have magnitude 7.6 units and 5.8 units respectively and the obtuse angle between the vectors is 118º, as shown in the diagram:



Find the magnitude of the vector $\overrightarrow{AC}$.

3 A particle is in equilibrium under the action of four horizontal forces of magnitudes 5 N, 8 N, P N and Q N, as shown in the diagram:



(a) Show that $P = 9$

(b) Find the value of Q.

F Exponentials and logarithms

F1 Know and use the function a^x and its graph, where a is positive. Know and use the function e^x and its graph.

Assessed at AS and A-level

Teaching guidance

Students should be able to:

- sketch and use simple transformations of the graph of the function a^x
- sketch and use simple transformations of the graph of the function e^x

Notes

- Simple transformations are defined in section B9.
- At A-level, students should also be able to sketch and use **a combination** of simple transformations of the functions a^x and e^x.

Examples

1 (a) Sketch the graph of $y = 3^x$, stating the coordinates of the point where the graph crosses the y-axis.

(b) Describe a single geometrical transformation that maps the graph of $y = 3^x$:

(i) onto the graph of $y = 3^{2x}$

(ii) onto the graph of $y = 3^{x+1}$

2 The curve $y = 3 \times 12^x$ is stretched by scale factor 2 parallel to the x-axis, then translated by the vector $\begin{pmatrix} 1 \\ p \end{pmatrix}$ to give the curve $y = \mathrm{f}(x)$

Given that the curve $y = \mathrm{f}(x)$ passes through the origin (0, 0), find the value of the constant p.

3 (a) Sketch the graph of $y = 9^x$, indicating the value of the intercept on the y-axis.

F2	Know that the gradient of $\mathbf{e}^{kx}$ is equal to $\boldsymbol{k}\mathbf{e}^{kx}$ and hence understand why the exponential model is suitable in many applications.

Assessed at AS and A-level

Teaching guidance

Notes

At AS, students should **know** that the gradient of Ae^{kx} is proportional to the value of the function.

At AS, students are **not** expected to differentiate functions involving e^{kx}

This is an unusual situation where the gradient is expected to be known without differentiation being formally used.

Students should understand that the exponential model is suitable in many applications because, if $y = e^{kx}$, $\frac{dy}{dx} = ky$ i.e. the rate of change of y with respect to x is proportional to y.

Examples

1 Find the gradient of the curve $y = e^{2x}$ at the point where $x = 2$

Circle your answer.

$2e$	$2e^2$	e^4	$2e^4$

2 A model for the growth of a colony of bacteria is $P = 500e^{\frac{1}{8}t}$, where P is the number of bacteria after t minutes.

What is the initial rate of growth of the bacteria?

Only assessed at A-level

Teaching guidance

Students should understand and be able to use exponential functions at a more in-depth level.

Example

1 A biologist is investigating the growth of a population of a species of rodent. The biologist proposes the model

$$N = \frac{500}{1 + 9e^{-\frac{t}{8}}}$$

for the number of rodents, N, in the population t weeks after the start of the investigation.

(a) Show that the rate of growth is given by

$$\frac{N}{4000}(500 - N)$$

(b) The maximum growth occurs after T weeks. Find the value of T

F3	Know and use the definition of $\log_a x$ as the inverse of a^x, where a is positive and $x \geq 0$. Know and use the function $\ln x$ and its graph. Know and use $\ln x$ as the inverse function of e^x

Assessed at AS and A-level

Teaching guidance

Students should:

- understand and be able to use the equivalences: $y = a^x \Leftrightarrow \log_a y = x$ and $y = e^x \Leftrightarrow \ln y = x$
- know that the graph of $y = \ln x$ is a reflection in the line $y = x$ of the graph of $y = e^x$
- be able to perform simple single transformations (as defined in section B9) of the functions $y = e^x$ and $y = \ln x$
- be able to manipulate logs and exponentials within the solution to a problem.

Examples

1 (a) Sketch the graph of $y = 2\ln x$ indicating any points where the curve crosses the coordinate axes.

(b) The graph of $y = 2\ln x$ can be transformed into the graph of $y = 1 + 2\ln x$ by means of a translation. Write down the vector of the translation.

2 If $A = B^n$, which of the following is true?

Circle your answer.

$n = \log_B A$ $\quad\quad$ $n = \log_A B$ $\quad\quad$ $B = \log_n A$ $\quad\quad$ $B = \log_A n$

3 Given that $\log_a b = c$, express b in terms of a and c

Only assessed at A-level

Teaching guidance

Students should be able to:

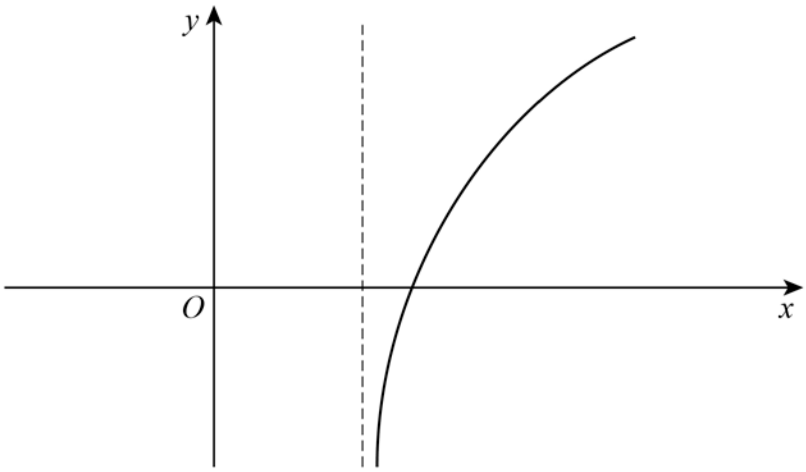
- sketch or use a combination of transformations of the functions $y = e^x$ and $y = \ln x$
- use the terms domain and range in problems using the functions $y = e^x$ and $y = \ln x$.

1 A function is defined by $f(x) = 2e^{3x} - 1$ for all real values of x

(a) Find the range of f

(b) Show that $f^{-1}(x) = \frac{1}{3}\ln\left(\frac{x+1}{2}\right)$

2 The curve with equation $y = f(x)$, where $f(x) = \ln(2x-3)$, $x > \frac{3}{2}$, is sketched below.



The inverse of f is f^{-1}

(a) Find $f^{-1}(x)$

(b) State the range of f^{-1}

(c) Sketch the curve with equation $y = f^{-1}(x)$, indicating the coordinates of the point where the curve intersects the y-axis.

F4	Understand and use the laws of logarithms: $\log_a x + \log_a y \equiv \log_a (xy)$; $\log_a x - \log_a y \equiv \log_a \left(\frac{x}{y}\right)$; $k \log_a x \equiv \log_a x^k$ (including, for example, $k = -1$ and $k = -\frac{1}{2}$)

Assessed at AS and A-level

Teaching guidance

Students should:

- know, understand and be able to use the above laws of logarithms
- know that $\log_a a = 1$ and $\log_a 1 = 0$ for $a > 0$

Examples

1 (a) Solve $3\log_a x = \log_a 8$

(b) Show that $3\log_a 6 - \log_a 8 = \log_a 27$

2 (a) Given that $\log_a x = 2\log_a 6 - \log_a 3$, show that $x = 12$

(b) Given that $\log_a y = \log_a 5 + 7$, express y in terms of a, giving your answer in a form not involving logarithms.

Only assessed at A-level

Example

1 The curve $y = 3^x$ intersects the line $y = x + 3$ at the point where $x = \alpha$

(a) Show that α lies between 0.5 and 1.5

(b) Show that the equation $3^x = x + 3$ can be arranged into the form

$$x = \frac{\ln(x+3)}{\ln 3}$$

F5 Solve equations of the form $a^x = b$.

Assessed at AS and A-level

Teaching guidance

Students should be able to solve equations of the form $a^x = b$, including $e^x = b$

Notes

- Equations of this form may require exact answers.
- If exact answers are not required such equations may be solved using a calculator, unless instructions are given to the contrary.

Examples

1 The line $y = 85$ intersects the curve $y = 6^{3x}$ at the point *A*. Find the x-coordinate of *A*, giving your answer to three decimal places.

2 The line $y = 21$ intersects the curve $y = 3(2^x + 1)$ at the point *P*

(a) Show that the x-coordinate of *P* satisfies the equation $2^x = 6$

(b) Find the x-coordinate of *P*, giving your answer to three significant figures.

3 Given that $e^{-2x} = 3$, find the exact value of x

F6

Use logarithmic graphs to estimate parameters in relationships of the form $y = ax^n$ and $y = kb^x$, given data for x and y.

Assessed at AS and A-level

Teaching guidance

Students should be able to:

- reduce a non-linear relationship to linear form
- plot a graph from given data, drawing a line of best fit by eye and using it to calculate the gradient and intercept to estimate for unknown constants.

Note: this is an essential skill in A-level sciences and there is an ideal opportunity here to link to real data: power laws for relationships of the form $y = ax^n$ and exponential laws for those of the form $y = kb^x$

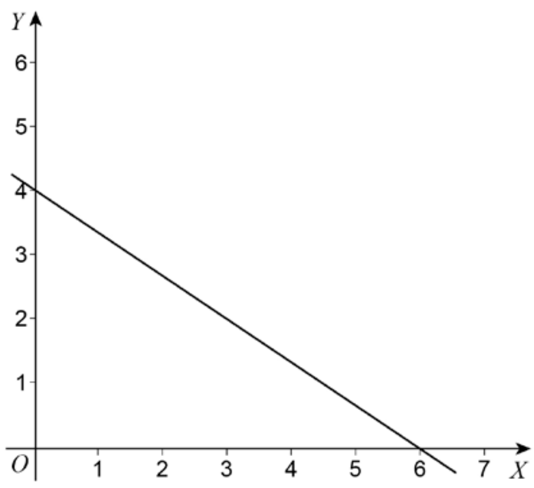
Examples

1 The variables y and x are related by an equation of the form $y = ax^n$ where a and n are constants.

Let $Y = \log_{10} y$ and $X = \log_{10} x$

(a) Show that there is a linear relationship between Y and X

(b) The graph of Y against X is shown in the diagram.



Find the value of n and the value of a

2 The variables x and y are known to be related by an equation of the form $y = ab^x$ where a and b are constants.

The following approximate values of x and y have been found.

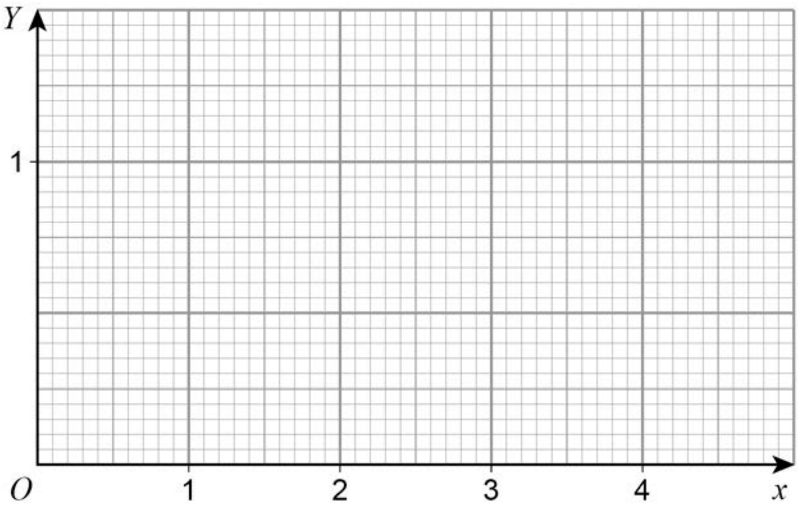
x	1	2	3	5
y	3.84	6.14	9.82	15.8

(a) Complete the table, showing values of x and Y, where $Y = \log_{10} y$. Give each value of Y to three decimal places.

x	1	2	3	4
Y	0.584			

(b) Show that, if $y = ab^x$, then x and Y must satisfy an equation of the form $Y = mx + c$

(c) Draw a graph relating x and Y



(d) Use a line of best fit on your graph to find estimates for the values of a and b

F7

Understand and use exponential growth and decay; use in modelling (examples may include the use of **e** in continuous compound interest, radioactive decay, drug concentration decay, exponential growth as a model for population growth); consideration of limitations and refinements of exponential models.

Assessed at AS and A-level

Teaching guidance

Students should be able to:

- use given conditions to determine the values of unknown constant terms in equations of the forms $y = Ae^{bx} + C$ or $P = Ak^t + C$
- form and use exponential equations to make predictions.

Examples

1 On 1 January 1900, a sculpture was valued at £80

When the sculpture was sold on 1 January 1956, its value was £5000

The value, £V, of the sculpture is modelled by the formula $V = Ak^t$, where t is the time in years since 1 January 1900 and A and k are constants.

(a) Write down the value of A

(b) Show that $k = 1.07664$ to five decimal places.

(c) Use this model to:

(i) show that the value of the sculpture on 1 January 2006 will be greater than £200 000

(ii) find the year in which the value of the sculpture will first exceed £800 000

(d) Explain whether your answer to (c)(ii) is, in reality, likely to be accurate.

2 A biologist is researching the growth of a certain species of hamster. She proposes that the length, x cm, of a hamster t days after its birth is given by $x = 15 - 12e^{-\frac{t}{14}}$

(a) Use this model to find:

(i) the length of a hamster when it is born.

(ii) the length of a hamster after 14 days, giving your answer to three significant figures.

(b) (i) Show that the time for a hamster to grow to 10 cm in length is given by

$$t = 14\ln\left(\frac{a}{b}\right)$$

where a and b are integers.

(ii) Find this time to the nearest day.

3 The concentration, C mg per litre of a particular drug in a patient's bloodstream t hours after it has been administered is given by the formula $C = C_0 e^{-0.2t}$

(a) Initially a patient is given a dose of 5 mg per litre.

(i) Write down the value of C_0

(ii) Find the concentration of the drug 3 hours after it is administered.

(b) The drug becomes ineffective when the concentration drops below 2 mg per litre

How long does it take for the drug to become ineffective? Give your answer to the nearest minute.

Note: within every set of exam papers at AS, 20% of the assessment must address AO3. At A-level, 25% of assessment addresses AO3.

Examples of modelling should be introduced to students at an early stage of teaching so that they can build confidence in the use of models and in the interpretation of the outputs from mathematical models. Models will not always be given to students. Students will sometimes be required to translate a situation in context into a mathematical model. Teachers should be mindful of both OT2 and OT3 because many assessment items will be set in the context of these overarching themes, which address problem-solving and modelling.

G Differentiation

G1

Understand and use the derivative of $\mathbf{f}(x)$ as the gradient of the tangent to the graph of $y = \mathbf{f}(x)$ at a general point (x, y); the gradient of the tangent as a limit; interpretation as a rate of change; sketching the gradient function for a given curve; second derivatives; differentiation from first principles for small positive integer powers of x and for $\sin x$ and $\cos x$

Understand and use the second derivative as the rate of change of gradient; connection to convex and concave sections of curves and points of inflection.

Assessed at AS and A-level

Teaching guidance

Students should be able to:

- recognise and use the notations $\mathrm{f}'(x)$, $\dfrac{\mathrm{d}y}{\mathrm{d}x}$, $\dfrac{\mathrm{d}}{\mathrm{d}x}(\mathrm{f}(x))$
- use $\mathrm{f}''(x)$ or $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$ to determine the nature of a stationary point. (See section G3)

Examples

1 A model helicopter takes off from a point O at time $t = 0$ and moves vertically so its height, y cm, above O after time t seconds is given by:

$$y = \frac{1}{4}t^4 - 26t^2 + 96t\,,\ 0 \le t \le 4$$

(a) Find:

(i) $\dfrac{\mathrm{d}y}{\mathrm{d}t}$

(ii) $\dfrac{\mathrm{d}^2y}{\mathrm{d}t^2}$

(b) Verify that y has a stationary value when $t = 2$ and determine whether this stationary value is a maximum value or a minimum value.

(c) Find the rate of change of y with respect to t when $t = 1$

(d) Determine whether the height of the helicopter above O is increasing or decreasing at the instant when $t = 3$

(e) Determine whether the speed of the helicopter is increasing or decreasing at the instant when $t = 3$

2 The volume, V m^3, of water in a tank at time t seconds is given by

$$V = \frac{1}{3}t^6 - 2t^4 + 3t^2 \text{ for } t \ge 0$$

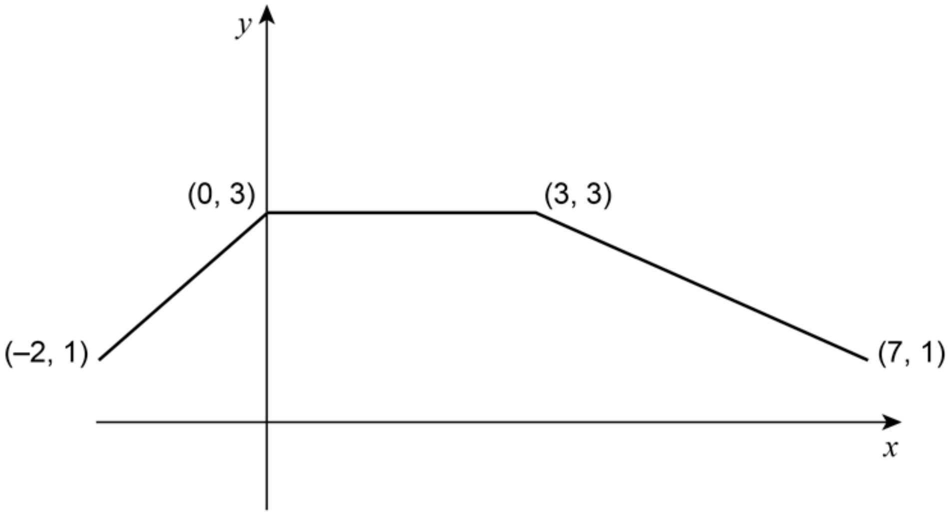
(a) Find:

(i) $\dfrac{\mathrm{d}V}{\mathrm{d}t}$

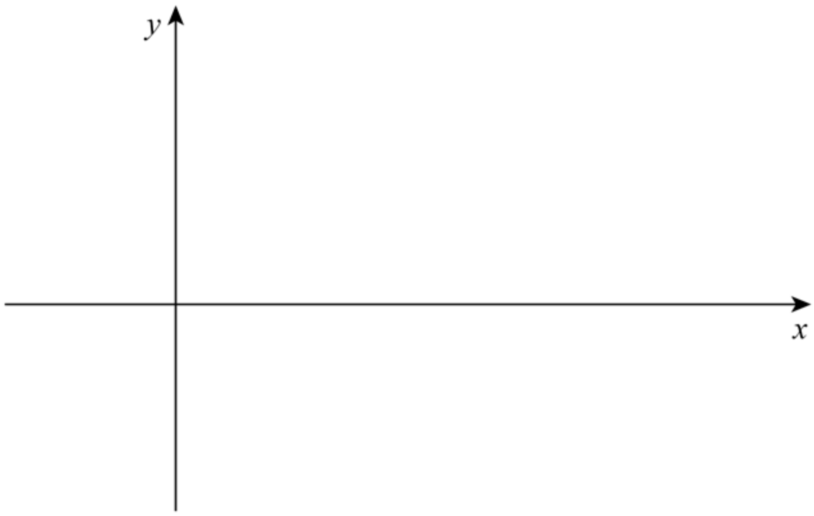
(ii) $\dfrac{\mathrm{d}^2V}{\mathrm{d}t^2}$

(b) Find the rate of change of the volume of water in the tank, in m^3 s^{-1}, when $t = 2$

3 The graph of $y = \mathrm{f}(x)$ is shown below.



Sketch the graph of $y = \mathrm{f}'(x)$ on the axes below.



4 A curve has equation $y = x^3 - 12x$

The point *A* on the curve has coordinates (2, –16)

The point *B* on the curve has x-coordinate $2 + h$

(a) Show that the gradient of the line *AB* is $6h + h^2$

(b) Explain how the result of part (a) can be used to show that *A* is a stationary point on the curve.

Only assessed at A-level

Teaching guidance

Students should:

- know that at points of inflection $f''(x)=0$, but that the converse is not necessarily true.
- know that for concave and convex sections of a function over the closed interval $[a, b]$ the following holds:
  - a twice-differentiable function is concave $\Leftrightarrow f''(x)\le 0$ for all $x\in[a, b]$
  - a twice-differentiable function is convex $\Leftrightarrow f''(x)\ge 0$ for all $x\in[a, b]$

Notes

- Students should verify the nature of a point of inflection, either by considering a change in concavity or by showing there is no change in sign for the first derivative.

Examples

1 A curve is given by the equation $f(x)=x^3-3x^2+1$

Find the range of values of x for which the curve is concave.

2 The point $A\left(\frac{\pi}{3}, \frac{1}{2}\right)$ is on the curve $y = \cos x$

The point B on the curve has x-coordinate $\frac{\pi}{3}+h$

(a) Show that the gradient of the line AB can be written as

$$\frac{\cos h-\sqrt{3}\sin h-1}{2h}$$

(b) Using the result from part (a) and small angle approximations explain how the gradient of the curve, $y = \cos x$, at the point A can be found.

G2	**Differentiate x^n, for rational values of n, and related constant multiples, sums and differences.** Differentiate e^{kx} and a^{kx}, sin kx, cos kx and tan kx, related sums, differences and constant multiples. Understand and use the derivative of ln x.

Assessed at AS and A-level

Teaching guidance

Students should know and be able to use the following:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^n\right) = nx^{n-1} \qquad \frac{\mathrm{d}}{\mathrm{d}x}\left(Ax^n\right) = Anx^{n-1} \qquad \frac{\mathrm{d}}{\mathrm{d}x}\left(Ax^n + Bx^m\right) = Anx^{n-1} + Bmx^{m-1}$$

Example

1 $x = \frac{1}{2}t^4 - 20t^2 + 66t$

Find:

(a) $\frac{\mathrm{d}x}{\mathrm{d}t}$

(b) $\frac{\mathrm{d}^2x}{\mathrm{d}t^2}$

Only assessed at A-level

Teaching guidance

Students should know and be able to use the following:

$\mathbf{f}(x)$	$\mathbf{f}'(x)$
e^{kx}	ke^{kx}
$\ln x$	$\frac{1}{x}$
$\sin kx$	$k\cos kx$
$\cos kx$	$-k\sin kx$
$\tan kx$	$k\sec^2 kx$
a^{kx}	$(k\ln a)a^{kx}$

The following is given in the formulae booklet:

$\tan x$	$\sec^2 x$

Examples

1 Find $\frac{dy}{dx}$ when $y = e^{3x} + \ln x$

2 A curve has the equation $y = e^{2x} - 10e^2 + 12x$

(a) Find $\frac{dy}{dx}$

(b) Find $\frac{d^2y}{dx^2}$

3 Find $\frac{dy}{dx}$ when $y = \tan 3x$

AQA

G3	Apply differentiation to find gradients, tangents and normals, maxima and minima and stationary points, points of inflection. Identify where functions are increasing or decreasing.

Assessed at AS and A-level

Teaching guidance

Students should:

- understand and be able to use the fact that at a stationary point, $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$
- describe a stationary point as a (local) maximum or minimum
- know that:

  At a maximum $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} < 0$

  At a minimum $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} > 0$

  Note:

  the case $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$ will not be tested at AS

- use $m_1 \times m_2 = -1$ for gradients of tangent and normal
- be able to answer questions set in the form of a practical problem where a function of a single variable has to be optimised
- be able to show that a function is increasing or decreasing, by showing $\frac{\mathrm{d}y}{\mathrm{d}x} > 0$ or $\frac{\mathrm{d}y}{\mathrm{d}x} < 0$ respectively.

Notes:

- There are different interpretations of increasing and decreasing functions. Whilst our preferred definition is that a function is increasing (decreasing) on an open interval (a, b) if $\frac{\mathrm{d}y}{\mathrm{d}x} > 0$ $\left(\frac{\mathrm{d}y}{\mathrm{d}x} < 0\right)$ then the definition that permits the derivative to equal zero is also acceptable.

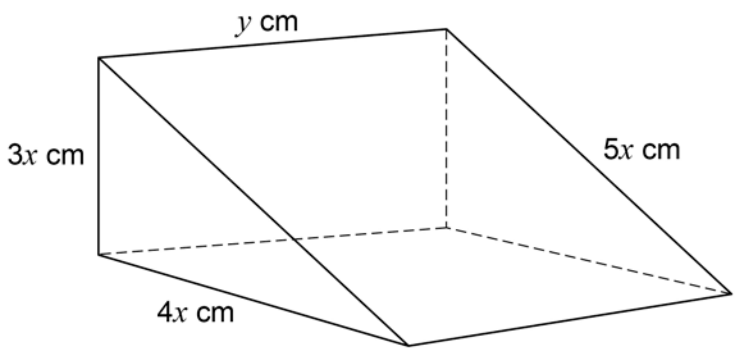
Examples

1 The curve with equation $y = x^4 - 32x + 5$ has a single stationary point, *M*

Find the coordinates of *M* and determine its nature.

Fully justify your answer.

2 The diagram shows a block of wood in the shape of a prism with a triangular cross-section. The end faces are right-angled triangles with sides of lengths $3x$ cm, $4x$ cm and $5x$ cm, as shown in the diagram.



The total surface area of the five faces is 144 cm^2.

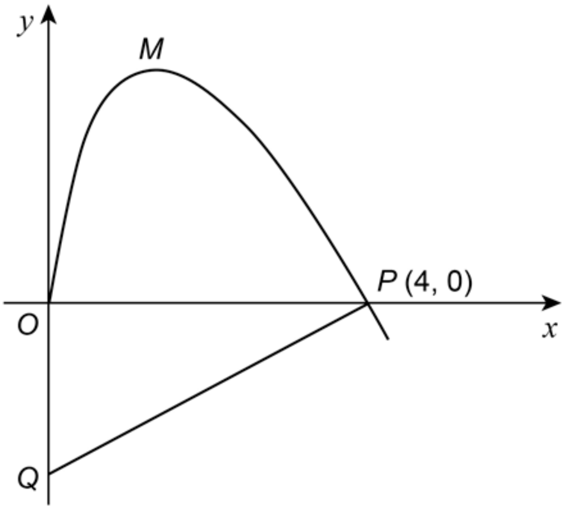
(a) Show that the volume of the block, $V\ \text{cm}^3$, is given by $V = 72x - 6x^3$

(b) Show that V has a stationary value when $x = 2$ and determine whether it is a maximum or a minimum.

Fully justify your answer.

3 A curve, drawn from the origin O, crosses the x-axis at the point $P(4, 0)$

The normal to the curve at P meets the y-axis at the point Q, as shown in the diagram.



The curve, defined for $x \geq 0$, has the equation $y = 4x^{\frac{1}{2}} - x^{\frac{3}{2}}$

(a) Find $\dfrac{dy}{dx}$

(b) Show that the gradient of the curve at P is -2

(d) Find the area of triangle OPQ

Only assessed at A-level

Teaching guidance

Students should:

- distinguish between stationary points and turning points
- know that a point of inflection is where a curve changes from convex to concave or vice versa and use this to test for points of inflection
- know that some points of inflection are stationary points, but that more often they are non-stationary points of inflection
- understand that $\frac{d^2y}{dx^2} = 0$ at a point of inflection, but this is not, on its own, sufficient to show the existence of a point of inflection, eg $y = x^4$ has a minimum at $x = 0$ (not a point of inflection) although $\frac{d^2y}{dx^2} = 0$ when $x = 0$
- be able to determine the nature of a stationary point when the second derivative is zero, by considering the sign of the first derivative either side of the stationary point.

Example

1 A curve has the equation $y = e^x + e^{-x}$

Show that any points of inflection of this curve satisfy $e^{2x} = -1$

2 A curve has the equation $y = e^{-2x} + 6x$

(a) Find the exact values of the coordinates of the stationary point of the curve.

(b) Determine the nature of the stationary point.

Fully justify your answer.

G4

Differentiate using the product rule, the quotient rule and the chain rule, including problems involving connected rates of change and inverse functions.

Only assessed at A-Level

Teaching guidance

Students should:

- know and be able to use the following rules:

Product rule	$\frac{\mathrm{d}}{\mathrm{d}x}\left[\mathrm{f}(x)\mathrm{g}(x)\right]=\mathrm{f}'(x)\mathrm{g}(x)+\mathrm{f}(x)\mathrm{g}'(x)$
Quotient rule (this is given in the formulae booklet)	$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\mathrm{f}(x)}{\mathrm{g}(x)}\right]=\frac{\mathrm{f}'(x)\mathrm{g}(x)-\mathrm{f}(x)\mathrm{g}'(x)}{\left[\mathrm{g}(x)\right]^2}$
Chain rule	If $y=\mathrm{f}(u)$ and $u=\mathrm{g}(x)$, so that $y=\mathrm{f}(\mathrm{g}(x))$ then $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}y}{\mathrm{d}u}\times\frac{\mathrm{d}u}{\mathrm{d}x}$
Inverses	$\frac{\mathrm{d}x}{\mathrm{d}y}=\frac{1}{\frac{\mathrm{d}y}{\mathrm{d}x}}$

- be able to use product, quotient or chain rules for differentiating $\tan x$, $\sec x$, $\operatorname{cosec} x$ and $\cot x$

Examples

1 (a) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ when:

(i) $y=\left(4x^2+3x+2\right)^{10}$

(ii) $y=x^2\tan x$

(b) (i) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ when $x=2y^3+\ln y$

(ii) Hence find an equation of the tangent to the curve $x=2y^3+\ln y$ at the point (2, 1)

2 (a) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ when:

(i) $y = (2x^2 - 5x + 1)^{20}$

(ii) $y = x\cos x$

(b) Given that

$$y = \frac{x^3}{x-2}$$

show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{kx^2(x-3)}{(x-2)^2}$$

where k is a positive integer.

3 A bucket is being used to catch water dripping from a leaking classroom ceiling, at a constant rate of 0.5 cm^3 per minute.

The volume, V cm^3, of water in the bucket is given by

$$V = \frac{\pi h}{3}\left(h^2 + 3h + 300\right)$$

Where h is the depth of water in the bucket.

Find the rate of change of h in cm per minute when the depth of water in the bucket is 10 cm.

G5	Differentiate simple functions and relations defined implicitly or parametrically, for first derivative only.

Only assessed at A-Level

Teaching guidance

Students should be able to:

- use the chain rule with parametric equations

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}y}{\mathrm{d}t}\times\frac{\mathrm{d}t}{\mathrm{d}x} \quad \text{or} \quad \frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}y}{\mathrm{d}t}\div\frac{\mathrm{d}x}{\mathrm{d}t}$$

- use the chain rule with parametric equations to find stationary points, equations of tangents and normals, but **not** to find points of inflection **nor** to test for concavity
- differentiate implicitly, using the product rule, quotient rule and chain rule as appropriate.

Examples

1 A curve is defined by the parametric equations

$$x = 1 + 2t, \quad y = 1 - 4t^2$$

(a) Hence find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of t

(b) Find an equation of the normal to the curve at the point where $t = 1$

2 A curve is defined by the equation

$$y^2 - xy + 3x^2 - 5 = 0$$

(a) Find the y-coordinates of the two points on the curve where $x = 1$

(b) (i) Show that $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{y-6x}{2y-x}$

(ii) Find the gradient of the curve at each of the points where $x = 1$

(iii) Show that, at the two stationary points on the curve, $33x^2 - 5 = 0$

3 A curve is defined by the parametric equations $x = 2\cos\theta$, $y = 3\sin 2\theta$

(a) (i) Show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = a\sin\theta + b\,\mathrm{cosec}\,\theta$$

where a and b are integers.

(ii) Find the gradient of the normal to the curve at the point where $\theta = \frac{\pi}{6}$

(b) Show that the Cartesian equation of the curve can be expressed as

$$y^2 = px^2(4 - x^2)$$

where p is a rational number.

4 A curve is defined by the equation $9x^2 - 6xy + 4y^2 = 3$

Find the coordinates of the two stationary points of this curve.

5 (a) A curve is defined by the equation $x^2 - y^2 = 8$

(i) Show that at any point (p, q) on the curve, where

$$q \neq 0$$

the gradient of the curve is given by

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{p}{q}$$

(ii) Show that the tangents at the points (p, q) and $(p, -q)$ intersect on the x-axis.

(b) Show that

$$x = t + \frac{2}{t}, \quad y = t - \frac{2}{t}$$

are parametric equations of the curve

$$x^2 - y^2 = 8$$

G6

Construct simple differential equations in pure mathematics and in context, (contexts may include kinematics, population growth and modelling the relationship between price and demand).

Only assessed at A-level

Teaching guidance

Students should understand language associated with proportionality and rates of change.

Examples

1 A water tank has a height of 2 metres. The depth of the water in the tank is h metres at time t minutes after water begins to enter the tank. The rate at which the depth of water in the tank increases is proportional to the difference between the height of the tank and the depth of the water.

Write down a differential equation in the variables h and t and a positive constant k

2 A biologist is investigating the growth of a population of a species of rodent. The biologist proposes the model

$$N = \frac{500}{1 + 9\mathrm{e}^{-\frac{t}{8}}}$$

for the number of rodents, N, in the population, t weeks after the start date of the investigation.

Use this model to answer the following questions.

(a) Show that the rate of growth is given by

$$\frac{N}{4000}(500 - N)$$

(b) The maximum rate of growth occurs after T weeks. Find the value of T

3 A giant snowball is melting. The snowball can be modelled as a sphere whose surface area is decreasing at a constant rate with respect to time. The surface area of the sphere is A cm^2 at time t days after it begins to melt.

Write down a differential equation in terms of the variables A and t and a constant k, where $k > 0$, to model the surface area of the melting snowball.

4 The number of fish in a lake is decreasing. After t years, there are x fish in the lake. The rate of decrease of the number of fish is proportional to the number of fish currently in the lake.

(a) Formulate a differential equation, in the variables x and t and a constant of proportionality k, where $k > 0$, to model the rate at which the number of fish in the lake is decreasing.

(b) At a certain time, there were 20 000 fish in the lake and the rate of decrease was 500 fish per year. Find the value of k

H Integration

H1 Know and use the Fundamental Theorem of Calculus.

Assessed at AS and A-level

Teaching guidance

Students should be able to:

- understand that differentiation is the 'reverse' of integration and vice versa
- use $\int_a^b \mathrm{f}(x)\,\mathrm{d}x = \mathrm{F}(b) - \mathrm{F}(a)$

  where $\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{F}(x)\right) = \mathrm{f}\left(x\right)$

  Note: the maximum level of difficulty for questions at AS requires students to use an integrand, $\mathrm{f}(x)$, where $\mathrm{f}(x)$ is the sum of terms of the form ax^n where n is rational and $n \neq -1$
- be able to find a function given its derivative and boundary condition.

Examples

1 The curve $y = \mathrm{f}(x)$ passes through (1, 3)

$\mathrm{f}'(x) = (2x - 3)(x + 1)^2$

Find $\mathrm{f}(x)$

2 $\mathrm{g}(x) = \frac{\mathrm{d}}{\mathrm{d}x}\int_1^x \left(t^2 - 3t\right)\mathrm{d}t$

Find $\mathrm{g}(3)$

H2

Integrate x^n (excluding n = –1), and related sums, differences and constant multiples.

Integrate e^{kx}, $\dfrac{1}{x}$, sin kx, cos kx and related sums, differences and constant multiples.

Assessed at AS and A-level

Teaching guidance

Students should:

- know that:

$$\int x^n \mathrm{d}x = \frac{x^{n+1}}{n+1} + c,\ n = -1$$

  and

$$\int ax^n + bx^m \mathrm{d}x = a\int x^n \mathrm{d}x + b\int x^m \mathrm{d}x$$

- understand integration as the reverse of differentiation
- include a constant of integration when finding an indefinite integral.

Examples

1 Find $\displaystyle\int \left(1 - \frac{1}{x^2}\right)^3 \mathrm{d}x$

2 Find $\displaystyle\int \left(1 + 3x^{\frac{1}{2}} + x^{\frac{3}{2}}\right) \mathrm{d}x$

Only assessed at A-level

Teaching guidance

Students should know and be able to use the following:

$\mathbf{f}(x)$	$\int \mathbf{f}(x)\,\mathbf{d}x$
e^{kx}	$\frac{1}{k}e^{kx} + c$
$\frac{1}{x}$	$\ln x + c$
$\sin kx$	$-\frac{1}{k}\cos kx + c$
$\cos kx$	$\frac{1}{k}\sin kx + c$

Notes

- Whilst it is correct to say that $\int \frac{1}{x}\,dx = \ln|x| + c$, the use of the modulus notation is not required at A-level.

Examples

1 Find $\int 2e^{4x} + \frac{1}{2x}\,dx$

2 (a) Find $\frac{dy}{dx}$ when $y = x\ln x$

(b) Hence find $\int \ln x\,dx$

3 Find $\int 4\sin 2x - 12\cos 3x\ dx$

H3

Evaluate definite integrals; use a definite integral to find the area under a curve and the area between two curves.

Assessed at AS and A-level

Teaching guidance

Students should be able to:

- understand and use the fact that for a function, f, where $\mathrm{f}(x) \geq 0$ for $a \leq x \leq b$ the area between the x-axis, the curve $y = \mathrm{f}(x)$ and the lines $x = a$ and $x = b$ is given by

$$\text{area} = \int_a^b \mathrm{f}\left(x\right)\mathrm{d}x$$

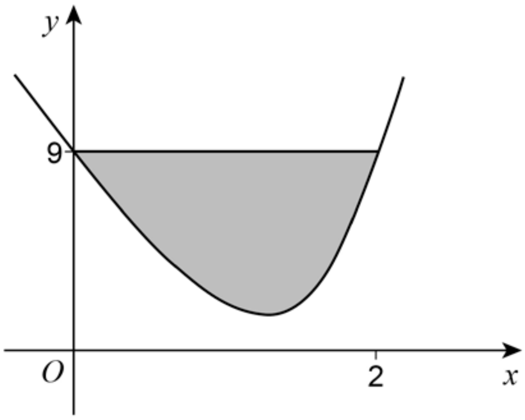
- understand that for areas lying **below** the x-axis the definite integral will give the negative of the required value
- find areas between curves and straight lines.

Notes

- Definite integrals can be found on a calculator and students are expected to do this in exams. If exact answers are required, these will usually require a non-calculator method.
- Students are **not** expected to find an area between a curve and the y-axis, by integrating an expression for x with respect to y

Examples

1 The curve with equation $y = x^4 - 8x + 9$ is sketched below.

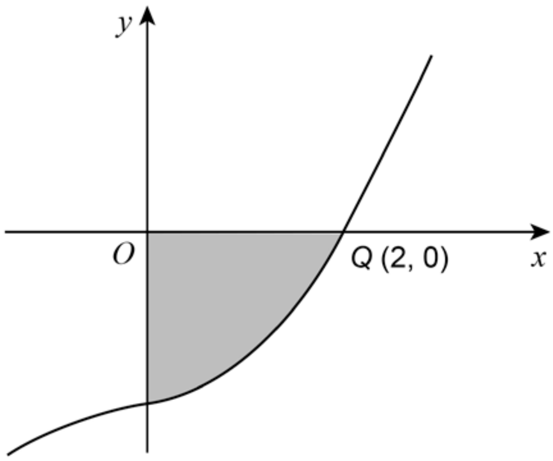


The point (2, 9) lies on the curve.

Find the area of the shaded region bounded by the curve and the line $y = 9$

Note: we would expect this question to be done on a calculator. The mathematical principles being tested in such a question are that the area under the curve can be found by integration and that the required area can be found by subtracting from the area of a rectangle.

2 The curve C with equation $y = x^3 + x - 10$, sketched below, crosses the x-axis at the point Q (2, 0)



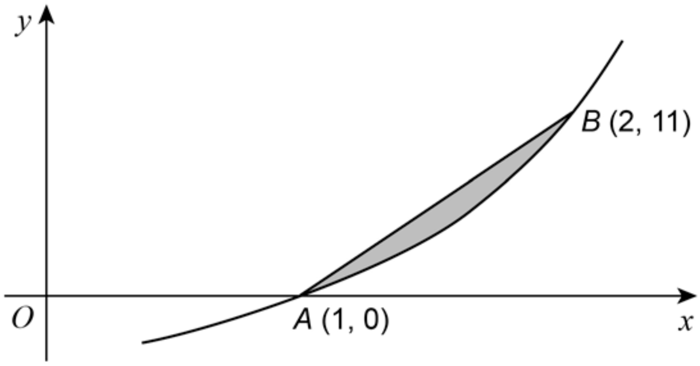
(a) Find an equation of the tangent to the curve C at the point Q

(b) Find $\int \left(x^3 + x - 10\right) \mathrm{d}x$

(c) Find the area of the shaded region bounded by the curve C and the coordinate axes.

Note: this part can be done using a calculator.

3 The curve with equation $y = x^3 + 4x - 5$ is sketched below.



The curve cuts the x-axis at the point $A(1, 0)$ and the point $B(2, 11)$ lies on the curve.

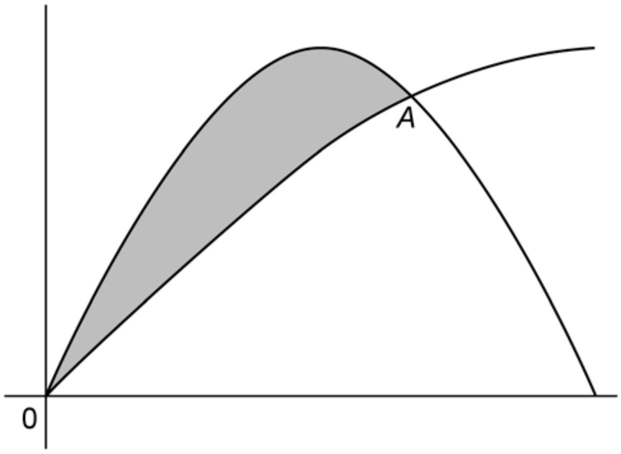
Find the area of the shaded region bounded by the curve and the line AB.

Note this can be considered as the area between two "curves," but the method expected is to subtract the area under the curve from the area of a triangle.

Only assessed at A-level

Example

1 The two curves, $y = \sin x$ and $y = \sin 2x$, for $0 \le x \le \frac{\pi}{2}$, are shown on the graph.



Find the exact area of the shaded region.

H4	Understand and use integration as the limit of a sum.

Only assessed at A-level

Teaching guidance

Students should be able to:

- understand that the area under a curve can be approximated using rectangles. The limit as the number of rectangles is increased is equal to a definite integral
- recognise and use notation such as $\lim_{n \to \infty} \sum_{i=1}^{n} y_i \delta x = \int_a^b y \, \mathrm{d}x$

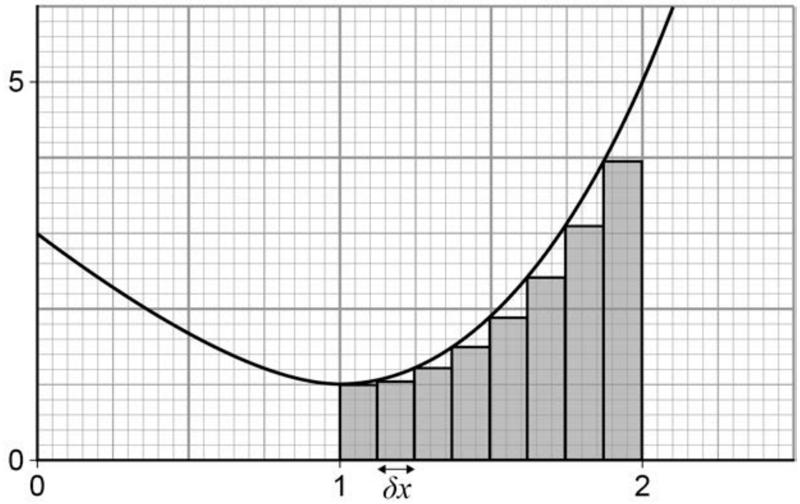
Notes: the idea here is to link to work at GCSE using the trapezium rule and to have a semi-formal understanding of the ideas of Riemann sums, although such vocabulary is not required. Students should comment on whether an approximation would give an over- or underestimate or if it cannot be decided, linking this to the ideas of increasing and decreasing functions.

The ideas here can be investigated using interactive applets such as intmath.com/integration/riemann-sums.php

This is an example of how the use of technology should permeate the course.

Example

1 The area under the graph of $y = x^3 - 3x + 3$ between the lines $x = 1$ and $x = 2$ is to be estimated by calculating the areas of the 8 rectangles, of equal width, δx, as shown in the diagram.



(a) What is the value of δx?

(b) The area of the smallest rectangle can be found using the calculation $y_1 \times \delta x$, where y_1 is the height of the rectangle.

Calculate $\sum_{i=1}^{8} y_i \delta x$

(c) A more accurate estimate of the area under the curve can be found by using more rectangles of smaller width.

Calculate $\lim_{n\to\infty} \sum_{i=1}^{n} y_i \delta x$

H5	Carry out simple cases of integration by substitution and integration by parts; understand these methods as the inverse processes of the chain and product rules respectively. (Integration by substitution includes finding a suitable substitution and is limited to cases where one substitution will lead to a function which can be integrated; integration by parts includes more than one application of the method but excludes reduction formulae.)

Only assessed at A-level

Teaching guidance

Students should be able to:

- understand the very simplest cases of substitution such as:

$$\int e^{ax+b}\mathrm{d}x$$

$$\int \sin(ax+b)\mathrm{d}x$$

$$\int \frac{1}{ax+b}\mathrm{d}x$$

Note: it is likely students will learn these as standard integrals rather than use substitution each time.

- recognise integrals of the form

$$\int f'(x)\cdot[f(x)]^n\mathrm{d}x$$

and integrate directly or by substitution

- use $\int u\frac{\mathrm{d}v}{\mathrm{d}x}\mathrm{d}x = uv - \int v\frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x$

to perform integration by parts, limited to a maximum of two consecutive applications of the method

- choose a simple substitution to perform an integral such as $\int \frac{x}{\sqrt{x+1}}\mathrm{d}x$ (This example is not exhaustive and students should have experience of various simple substitutions.)
- integrate curves given in parametric form and find the area under a curve given in parametric form.

Examples

1 Find $\int x^2 \sin 2x \, \mathrm{d}x$

2 Use the substitution $x = \sin\theta$ to find

$$\int \frac{1}{\left(1-x^2\right)^{\frac{3}{2}}} \mathrm{d}x$$

giving your answer in terms of x

3 (a) (i) Find $\int \ln x \, \mathrm{d}x$

(ii) Find $\int (\ln x)^2 \, \mathrm{d}x$

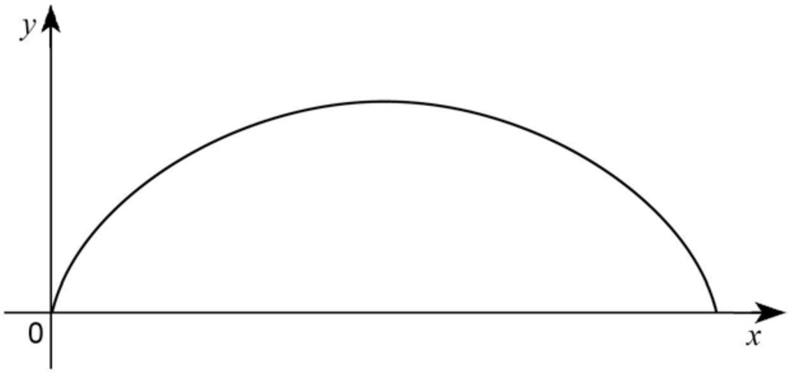
(b) Use the substitution $u = \sqrt{x}$ to find the exact value of $\int_1^4 \frac{1}{x+\sqrt{x}} \mathrm{d}x$

4 Find $\int x^2 \left(x^3 + 2\right)^7 \mathrm{d}x$

5 Find $\int \frac{x^2 + 2x}{2x^3 + 6x^2 + 1} \mathrm{d}x$

6 Find $\int \frac{\sin x}{\cos x} \mathrm{d}x$

7 The curve with parametric equations $x = 1 + \cos\theta$ and $y = \sin\theta, 0 \le \theta \le \pi$ is shown below.

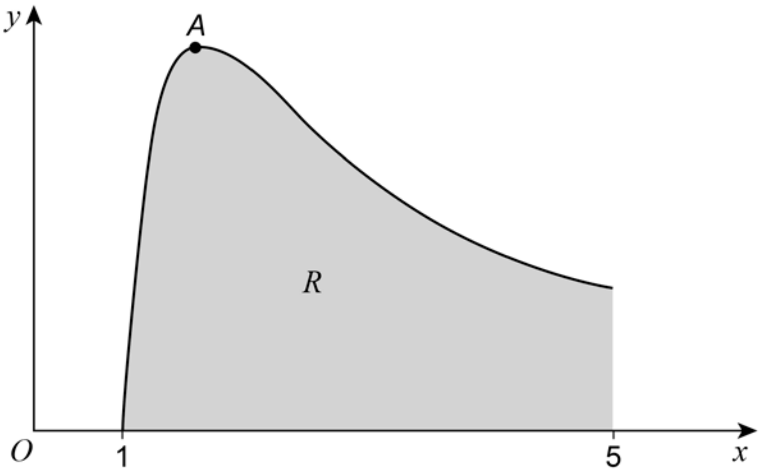


Find the area enclosed between the curve and the x-axis.

8 (a) Given that $y = x^{-2}\ln x$, show that $\dfrac{dy}{dx} = \dfrac{1 - 2\ln x}{x^3}$

(b) Using integration by parts, find $\int x^{-2}\ln x \, dx$

(c) The sketch shows the graph of $y = x^{-2}\ln x$



Find the exact value of the shaded area R.

H6

Integrate using partial fractions that are linear in the denominator.

Only assessed at A-level

Teaching guidance

Students should be able to answer questions that require the simplification of a more complicated fraction, which leads to an integrable expression or to partial fractions that can be integrated.

Examples

1 (a) Find $\displaystyle\int \frac{5x-6}{x(x-3)}\,\mathrm{d}x$

(b) (i)

$$4x^3 + 5x - 2 \equiv (2x+1)(2x^2 + px + q) + r$$

Find the values of the constants p, q and r

(ii) Hence, find $\displaystyle\int \frac{4x^3+5x-2}{2x+1}\,\mathrm{d}x$

2 (a) (i) Express

$$\frac{5-8x}{(2+x)(1-3x)}$$

in the form

$$\frac{A}{2+x}+\frac{B}{1-3x}$$

where A and B are integers.

(ii) Hence show that

$$\int_{-1}^{0}\frac{5-8x}{(2+x)(1-3x)}\,\mathrm{d}x = p\ln 2$$

where p is rational.

H7

Evaluate the analytical solution of simple first order differential equations with separable variables, including finding particular solutions.

(Separation of variables may require factorisation involving a common factor.)

Only assessed at A-level

Teaching guidance

Students should be able to carry out any of the techniques of integration included in sections H1 to H6.

Examples

1 Solve the differential equation

$$\frac{dy}{dx} = 6xy^2$$

given that $y = 1$ when $x = 2$

Give your answer in the form $y = f(x)$

2 Solve the differential equation

$$\frac{dy}{dx} = \frac{1}{y}\cos\left(\frac{x}{3}\right)$$

given that $y = 1$ when $x = \frac{\pi}{2}$

Write your answer in the form $y^2 = f(x)$

3 (a) Solve the differential equation

$$\frac{dy}{dt} = y\sin t$$

to obtain y in terms of t

(b) Given that $y = 50$ when $t = \pi$, show that $y = 50e^{-(1+\cos t)}$

4 Find the general solution of the differential equation

$$\frac{dy}{dx} = xy + 2x$$

Give your answer in the form $y = f(x)$

5 Solve the differential equation

$$\frac{dy}{dx} = \frac{x\sqrt{x^2+3}}{e^{2y}}$$

given that $y = 0$ when $x = 1$

Give your answer in the form $y = f(x)$

H8

Interpret the solution of a differential equation in the context of solving a problem, including identifying limitations of the solution; includes links to kinematics.

Only assessed at A-level

Teaching guidance

Students should be able to solve differential equations they have set up themselves.

Examples

1 A car's value **depreciates** at a rate which is proportional to its value, £V, at time t months from when it was new.

(a) Write down a differential equation in terms of the variables V and t and a constant k, where $k > 0$, to model the value of the car.

(b) Solve your differential equation to show that $V = Ad^t$ where $d = e^{-k}$

(c) The value of the car when new was £12 499 and 36 months later its value was £7000

Find the values of A and d

2 The platform of a theme park ride oscillates vertically. For the first 75 seconds of the ride

$$\frac{dx}{dt} = \frac{t\cos\left(\frac{\pi}{4}t\right)}{32x}$$

where x metres is the height of the platform above the ground after time t seconds.

At $t = 0$, the height of the platform above the ground is 4 metres.

Find the height of the platform after 45 seconds, giving your answer to the nearest centimetre.

I Numerical methods

I1	Locate roots of $f(x) = 0$ by considering changes of sign of $f(x)$ in an interval of x on which $f(x)$ is sufficiently well-behaved. Understand how change of sign methods can fail.

Only assessed at A-level

Teaching guidance

Students should:

- be able to answer questions where roots of equations in the form $g(x) = h(x)$ can be located by rearranging to give $f(x) = g(x) - h(x) = 0$
- know that discontinuities can cause failure but also know that the converse is not true
- clearly state that a curve is continuous when concluding that a root exists in a given interval.

Examples

1 The curve $y = 3^x$ intersects the line $y = x + 3$ at the point where $x = \alpha$

Show, using the change of sign method, that α lies between 0.5 and 1.5

2 Show, using the change of sign method, that the equation

$$x^3 - 6x + 1 = 0$$

has a root α, where

$$2 < \alpha < 3$$

3 When searching for a root to the equation $f(x) = 0$, a student correctly evaluates $f(1) = 4$ and $f(2) = -3$

Given that f is a continuous function, which one of the following could not be true?

Circle the correct answer.

There are no roots for $1 < x < 2$	There is one root for $1 < x < 2$	There are two roots for $1 < x < 2$	There are three roots for $1 < x < 2$

I2	Solve equations approximately using simple iterative methods; be able to draw associated cobweb and staircase diagrams. Solve equations using the Newton-Raphson method and other recurrence relations of the form $x_{n+1} = \mathrm{g}(x_n)$. Understand how such methods can fail.

Only assessed at A-level

Teaching guidance

Students should be able to:

- answer questions that require them to rearrange an equation into an iterative form
- draw staircase and cobweb diagrams, to illustrate the iteration, on printed graphs
- demonstrate divergence on a diagram
- understand the conditions when the Newton-Raphson method may fail.

Notes

- This is an opportunity to use technology. Interactive software, such as GeoGebra, allows you to investigate cobweb diagrams: geogebra.org/m/XvjM7Xnv
- Students must use calculators to perform iterative operations.

Examples

1 (a) The equation $e^{-x} - 2 + \sqrt{x} = 0$ has a single root, α

Show that α lies between 3 and 4

(b) Use the recurrence relation

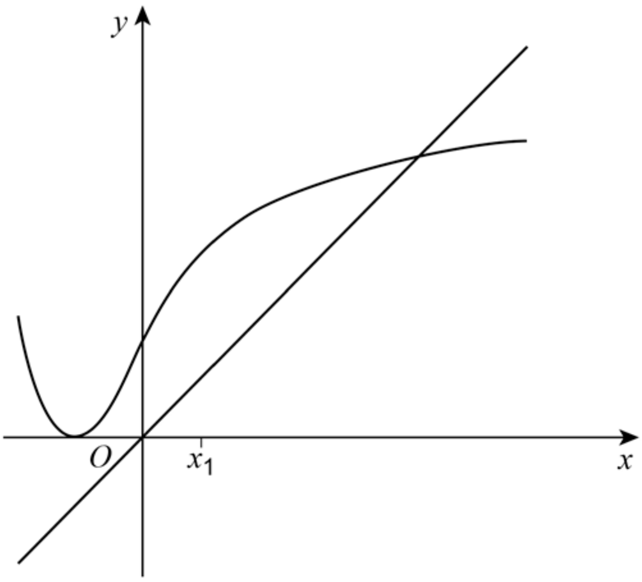
$$x_{n+1} = \left(2 - e^{-x_n}\right)^2$$

with

$$x_1 = 3.5$$

to find x_2 and x_3 giving your answers to three decimal places.

(c) The diagram shows parts of the graphs of $y = \left(2 - e^{-x}\right)^2$ and $y = x$, and a position of x_1



On the diagram, draw a staircase or cobweb diagram to show how convergence takes place, indicating the positions of x_2 and x_3 on the x-axis.

2 The equation $24x^3 + 36x^2 + 18x - 5 = 0$ has one real root, α

(a) Show that α lies in the interval $0.1 < x < 0.2$

(b) Taking $x_1 = 0.2$ as a first approximation to α, use the Newton-Raphson method to find a second approximation, x_2, to α. Give your answer to four decimal places.

3 The equation $x^3 - x^2 - 2 = 0$ has a single root between 1 and 2

Starting with $x_1 = 1$ which of the following iterative formulae will not converge on the root?

Circle your answer.

$x_{n+1} = \sqrt{\dfrac{(x_n)^2 + 2}{x_n}}$ $\qquad$ $x_{n+1} = \sqrt[3]{(x_n)^2 + 2}$ $\qquad$ $x_{n+1} = \dfrac{2}{(x_n)^2} + 1$ $\qquad$ $x_{n+1} = \dfrac{(x_n)^3 - 2}{x_n}$

4 A curve has equation $y = x^3 - 3x + 3$

(a) Show that the curve intersects the x-axis at the point $(\alpha, 0)$ where $-3 < \alpha < -2$

(b) A student attempts to find α using the Newton-Raphson method with $x_1 = -1$

Explain why the student's method fails.

5 Use the iteration

$$x_{n+1} = \frac{\ln(x_n + 3)}{\ln 3}$$

with

$$x_1 = 0.5$$

to find x_3 to two significant figures.

I3	Understand and use numerical integration of functions, including the use of the trapezium rule and estimating the approximate area under a curve and limits that it must lie between.

Only assessed at A-level

Teaching guidance

Students should be able to:

- understand and use the term ‘ordinate’
- use graphical determination to find whether an approximation over- or underestimates the area, depending on the concavity of the curve
- improve an approximation by increasing the number of ordinates or strips used.

This topic links to H4 and can also be usefully explored using interactives.

Examples

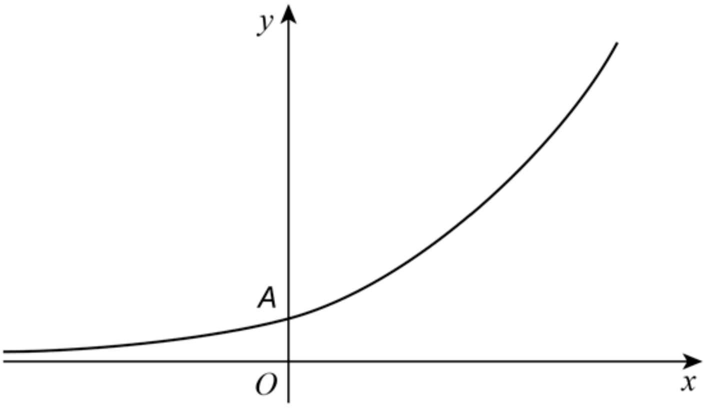
1 (a) Use the trapezium rule with five ordinates to find an approximate value for

$$\int_0^4 \frac{1}{x^2+1}\,\mathrm{d}x$$

giving your answer to four significant figures.

(b) State how you could obtain a better approximation to the value of the integral using the trapezium rule.

2 The diagram shows a sketch of the curve $y = 2^{4x}$



The curve intersects the y-axis at the point *A.*

(a) Find the value of the y-coordinate of *A.*

(b) Use the trapezium rule with six ordinates to find an approximate value for

$$\int_0^1 2^{4x}\,\mathrm{d}x$$

giving your answer to two decimal places.

3 The trapezium rule is used, with six ordinates, to find an estimate for the value of

$$\int_0^5 \mathrm{f}(x)\,\mathrm{d}x$$

Given that f is an increasing function and $\mathrm{f}''(x) > 0$, explain what would happen to the value obtained by the trapezium rule if the number of ordinates was increased.

AQA

14 Use numerical methods to solve problems in context.

Only assessed at A-level

Teaching guidance

Students should be able to:

- use a numerical method to solve an equation or to calculate an area resulting from a problem that has its origin within pure mathematics or from a context such as exponential growth or kinematics
- demonstrate that they can apply the correct numerical method in full. Numerical solutions obtained directly from calculator functions are unlikely to achieve full credit.

Examples

1 A particle travels in a straight line with its velocity, v m s^{-1}, at time, t seconds, given by

$$v = 10e^{\frac{t}{5}}\sin t$$

Use the trapezium rule, with 6 ordinates, to estimate the distance moved by the particle in its first 5 seconds of motion.

2 A particle is projected vertically upwards so that its height, h metres, above the ground after time t seconds is given by

$$h = 20t - 9.8t^2 + 2.5e^{-t}$$

(a) How high above the ground was the point where the particle was initially launched?

(b) Given that the particle is in the air for between 1.5 and 2.5 seconds, use the Newton-Raphson method to find the time when the particle hits the ground. Give your answer to two significant figures.

Note: Teachers should remind students of OT2.4: many mathematical problems cannot be solved analytically, but numerical methods permit a solution to a required level of accuracy. This is a subtle and very important point: students are often comfortable with what they consider trial and error methods; we teach them analytical methods that are generally more efficient, but now we are saying that sometimes there are problems that require numerical methods.

J Vectors

J1 Use vectors in two dimensions and in three dimensions.

Assessed at AS and A-level

Teaching guidance

Students should:

- become familiar with both column vectors and **i**, **j** notation, where **i** and **j** are unit vectors in perpendicular directions. Unless specified otherwise, students may use either notation in their solutions

  Note: questions may be set without context as pure vector questions, or in context, for example, using vectors to represent velocities or forces.

- know that vectors may be used to describe translations of graphs.

Notes

- We will use square brackets for column vectors in exam questions, but round brackets are equally acceptable.
- When writing vectors using single lower-case letters, students may underline or overline, but no particular notation will be required.
- For the purposes of examinations at AS and A-level the topic of Vectors is treated as part of Mechanics and will only be found in the Mechanics section of AS Paper 1 and of A-level Paper 2.
- Teachers will note that the vector content of A-level Maths has considerably reduced in the new specification.

Examples

1 The points A and B have coordinates (–2, 3) and (4, –1) respectively.

Write down the vector $\overrightarrow{AB}$ in the form $a\mathbf{i} + b\mathbf{j}$

2 A force, **F**, acts on a particle of mass 2 kg.

Given that $\mathbf{F} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ N, find the acceleration of the particle.

Only assessed at A-level

Teaching guidance

Students should become familiar with both column vectors and **i**, **j**, **k** notation, where **i**, **j** and **k** are unit vectors in mutually perpendicular directions in a right-handed coordinate system. Unless specified otherwise, students may use either notation in their solutions.

Example

1 The points A and B have coordinates (–2, 3, 1) and (4, –1, –2) respectively.

Write down the vector $\overrightarrow{AB}$ in the form $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$

J2

Calculate the magnitude and direction of a vector and convert between component form and magnitude/direction form.

Assessed at AS and A-level

Teaching guidance

Students should be able to answer questions using vectors in two dimensions.

Examples

1 Calculate the angle between the vector **j** and the vector 2**i** + 3**j**

2 A cricket ball is hit at ground level on a horizontal surface.

It initially moves at 26 m s^{-1} at an angle of 50° above the horizontal.

(a) Find the horizontal component of the ball's initial velocity, giving your answer to two significant figures.

(b) Write down the initial velocity of the ball as a vector in the form a**i** + b**j**

3 Two forces, **P** = (6**i** – 3**j**) N and **Q** = (3**i** + 15**j**) N, act on a particle.

The unit vectors **i** and **j** are perpendicular.

(a) State the resultant of **P** and **Q.**

(b) Calculate the magnitude of the resultant of **P** and **Q.**

(c) Calculate the direction of the resultant of **P** and **Q.**

Note: when answering part (c), students would be expected to be specific about where they are measuring their direction from. This can be done with a clear diagram or described as 53.1° from the **i** direction, for example.

4 A ship is sailing with a constant velocity of $\begin{bmatrix} 5 \\ -7 \end{bmatrix}$ km/h, where the horizontal component of the vector is east and the vertical north.

(a) Find the speed of the ship.

(b) Find the direction in which the ship is sailing, giving your answer as a 3-figure bearing to the nearest degree.

Only assessed at A-level

Teaching guidance

Students should be able to answer questions using vectors in two or three dimensions.

Example

1 The points A and B have coordinates (2, 5, 1) and (4, 1, –2) respectively.

(a) Find the vector $\overrightarrow{AB}$

(b) Find the length AB

J3

Add vectors diagrammatically and perform the algebraic operations of vector addition and multiplication by scalars, and understand their geometrical interpretations.

Assessed at AS and A-level

Teaching guidance

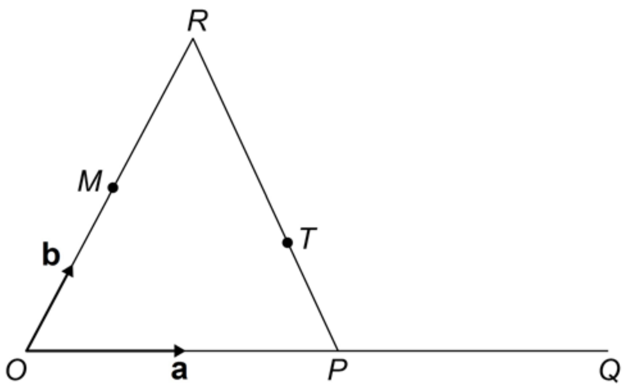
Students should be able to:

- prove that two vectors are parallel to each other by showing that one is a multiple of the other
- recall the conditions for collinearity
- understand that a vector diagram can be used to find resultants. This could, for example, be in the context of force.

Note: unless specifically stated in a question, students can choose **any** appropriate method to solve vector problems.

Examples

1



$\overrightarrow{OP} = \overrightarrow{PQ} = \mathbf{a}$

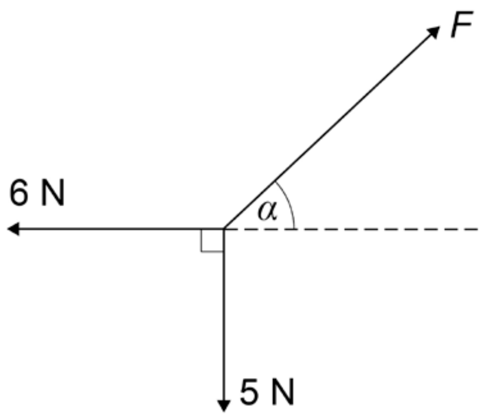
$\overrightarrow{OM} = \overrightarrow{MR} = \mathbf{b}$

The point T is on the line PR such that PT:TR = 1:2

(a) Find $\overrightarrow{PT}$ in terms of **a** and **b**.

(b) Prove that M, T and Q are collinear.

2 The diagram shows three forces which act in the same plane and are in equilibrium.



(a) Find F

(b) Find α

Only assessed at A-level

Example

1 The points A, B and C have coordinates (3, 1, –6), (5, –2, 0) and (8, –4, –6) respectively.

(a) Show that the vector $\overrightarrow{AC}$ is given by $\overrightarrow{AC} = n\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, where n is an integer.

(b) The point D is such that $ABCD$ is a rhombus.

(i) Find the coordinates of D.

(ii) Calculate the side length of the rhombus.

J4 Understand and use position vectors; calculate the distance between two points represented by position vectors.

Assessed at AS and A-level

Teaching guidance

Students should be able to:

- understand and use the result $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$
- use position vectors in both a pure context and in mechanics questions, for example in kinematics.

Example

1 The points A and B have position vectors $\overrightarrow{OA} = 5\mathbf{i} - 2\mathbf{j}$ and $\overrightarrow{OB} = -\mathbf{i} + 3\mathbf{j}$

Find the distance between the points A and B.

Only assessed at A-level

Examples

1 A particle is initially at the point A, which has position vector $13.6\mathbf{i}$ metres, with respect to an origin O

At the point A, the particle has velocity $(6.0\mathbf{i} + 2.4\mathbf{j})$ m s^{-1} and in its subsequent motion, it has a constant acceleration of $(-0.80\mathbf{i} + 0.10\mathbf{j})$ m s^{-2}

The unit vectors $\mathbf{i}$ and $\mathbf{j}$ are directed east and north respectively.

Find an expression for the position vector of the particle, with respect to the origin O, t seconds after it leaves A.

2 The positions of two particles, A and B, at a time t, are given by $\mathbf{r}_A = 3\cos(\pi t)\mathbf{i} + (2t+1)\mathbf{j}$ and $\mathbf{r}_B = 2\sin(\pi t)\mathbf{i} + (2-t)\mathbf{j}$ respectively.

Find the distance between the two particles when $t = 4$

AQA

J5 Use vectors to solve problems in pure mathematics and in context, including forces and kinematics.

Assessed at AS and A-level

Teaching guidance

Students should be able to use vectors in 2 dimensions to solve problems.

Note: at AS, questions on kinematics will **not** involve vectors but questions related to forces may involve 2D vectors.

Example

1 Three forces $\mathbf{F}_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ N, $\mathbf{F}_2 = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$ N and $\mathbf{F}_3 = \begin{bmatrix} a \\ b \end{bmatrix}$ N act at a point and are in equilibrium.

Find a and b.

Only assessed at A-level

Teaching guidance

Students should be able to:

- use vectors in up to **3** dimensions to solve problems
- use vectors to solve problems related to kinematics in up to **2** dimensions, including projectile motion.

Examples

1 Three points A, B and C are collinear with $\overrightarrow{OA} = \begin{bmatrix} 1 \\ 5 \\ -2 \end{bmatrix}$ and $\overrightarrow{OB} = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$

Given that $AC = 2AB$ find the coordinates of the two possible positions of point C

2 The velocity of a particle is $\begin{bmatrix} 2t^2 - 2 \\ 3t + 1 \end{bmatrix}$ m s^{-1}

(a) Show that the particle is never at rest.

(b) Find the time when the magnitude of the acceleration is 1.7 m s^{-2}

3 In this question use g = 10 m s^{-2}

A particle is projected from the point with coordinates (2, 1) with an initial velocity of $\begin{bmatrix} 7 \\ 2 \end{bmatrix}$ m s^{-1} and moves freely under gravity.

(a) Show that the position vector of the particle t seconds after it is projected is given by $\mathbf{r} = \begin{bmatrix} 2 + 7t \\ 1 + 2t - 5t^2 \end{bmatrix}$ m

(b) Find the distance of the particle from its starting point when it reaches its maximum height.

For sections K to O students must use a calculator that can find:

- exact and cumulative probabilities in the binomial distribution
- probabilities in a normal distribution.

Teachers are advised to teach as much as possible within the context of the large data set that is current within the specification.

K Statistical sampling

K1	Understand the terms ‘population’ and ‘sample’. Use samples to make informal inferences about the population. Understand and use sampling techniques, including simple random sampling and opportunity sampling. Select or critique sampling techniques in the context of solving a statistical problem, including understanding that different samples can lead to different conclusions about the population.

Assessed at AS and A-level

Teaching guidance

Students should be able to:

- identify a population and understand and explain the meaning of the terms parameter and statistic
- understand that information from a sample can be used to make inferences about the population it was taken from, so that a sample statistic can be used to estimate a population parameter
- recognise and explain how to use sampling techniques, to include simple random sampling, systematic sampling, opportunity sampling, stratified sampling, quota sampling and cluster sampling
- understand that the use of prior information can make the sample more representative of the population (eg stratification)
- decide which sampling method to use to overcome practical problems when sampling
- discuss the advantages and disadvantages of each sampling method within a given context

- understand that increasing the sample size gives more reliable information about a population.

  Note: formal treatment of the estimation of population parameters by sample statistics is not required.

Examples

1 A large number of spectators attend a football match at a ground in England. One of the stands contains 1390 seats, numbered from 1 to 1390

(a) (i) Describe how random numbers could be used to select a random sample of 80 of these seats.

(ii) The ground authority wishes to carry out a survey into spectators' opinions of the catering facilities at the ground. It proposes to ask the spectators occupying the 80 seats selected in part (a) (i) to complete a questionnaire.

Suggest **two** practical difficulties that might be encountered in carrying out this proposal.

(b) A market research company is employed to investigate the amount of interest taken in sport by the population of the United Kingdom. James, a new employee of the company, suggests that interviewers should be stationed outside each exit from the ground with instructions to interview every 100th person leaving the ground after the match.

(i) State the name given to this type of sampling.

(ii) Give **one** reason why the results of the investigation suggested by James might be biased.

(iii) Suggest **two** practical difficulties, other than those which you mentioned in part (a) (ii), which might be encountered in carrying out these interviews.

2 A fan club has 2076 members who are divided geographically into eight branches. The club's committee wishes to seek members' views on where to hold the next annual meeting. A sample of 100 members is to be obtained and their views sought.

The following suggestions are made as to how to choose the sample.

Suggestion A Members are selected from all eight branches. The number of members from each branch is proportional to the size of the branch. The branch secretaries are asked to choose the appropriate number of members from their branches in any convenient way.

Suggestion B Two of the branches are selected at random. Fifty members from each of these branches are selected at random.

Suggestion C Members are selected by a random process from each branch. The number of members from each branch is proportional to the size of the branch.

Suggestion D Members are selected by the chair of the committee, from the branch she is a member of, at the next local meeting.

(a) For **each** of the four suggestions:

(i) Name the method of sampling.

(ii) Either state that each member is equally likely to be included in the sample, or explain why this is not the case.

(b) (i) State, giving a reason, which of the four methods is preferable from a statistical point of view.

(ii) Give a reason why Suggestion A might be preferred to Suggestion C.

3 Emile wants to investigate the proportion of the population who watch soap operas on television.

One day, he asks each of the students in his drama class: "Did you watch a soap opera on television last night?"

(a) Name the type of sampling that Emile is using.

(b) Comment on the use of the proportion of students who watched a soap opera last night, obtained from Emile's sample, to be representative of the proportion of the population who watch soap operas on television.

Note: part (b) of this question does not specify exactly what a student is required to include in their response. This leaves it to the student to decide on their own response. Possible responses to this may be along the following lines:

Emile's opportunity sample is reflecting only a very restricted sample of the population, namely only students/drama students in his class at one school/college in the country.

Not every member of the population under investigation is equally likely to be included in the sample so the results are likely to contain bias.

It is unlikely that the proportion of students who watched a soap opera last night will be representative of the proportion of the population who watch soap operas.

4 Packets of a particular type of sweet are known to have a mean of 100 grams. The number of sweets in a packet is approximately 30 and the sweets come in any one of five flavours. The weights of 50 packets are taken and the mean is found to be 98.3 grams.

From the above passage, identify:

(a) a population

(b) a parameter

(c) a sample

(d) a qualitative variable

(e) a continuous variable

(f) a discrete variable

5 Hermione wants to investigate the proportion of people in her school who enjoy watching horror movies.

She decides to go from table to table in the school refectory and to ask the question ‘Do you enjoy watching horror movies?’ to each table as a group.

She will record the number of ‘yes’ responses and the total number of students for each table.

Hermione assumes that each student will respond independently to her question.

(a) Comment on the assumption that each student will respond independently to her question.

(b) Suggest how Hermione could change her method of collecting data to improve the reliability of her data.

Note: this example asks students to consider how more reliable data may be achieved. This may be achieved, for example, by getting each student to record their response on a piece of paper and folding this before putting it into a box so that no other student sees their response (anonymity being more likely to elicit a genuine response rather than one influenced by other students on the table.) Students may also suggest that respondents could be asked to complete their responses online where there is no mention of who is conducting the investigation and that this may then reduce possible bias in responses given etc.

L Data presentation and interpretation

L1

Interpret diagrams for single-variable data, including understanding that area in a histogram represents frequency.

Connect to probability distributions.

Assessed at AS and A-level

Teaching guidance

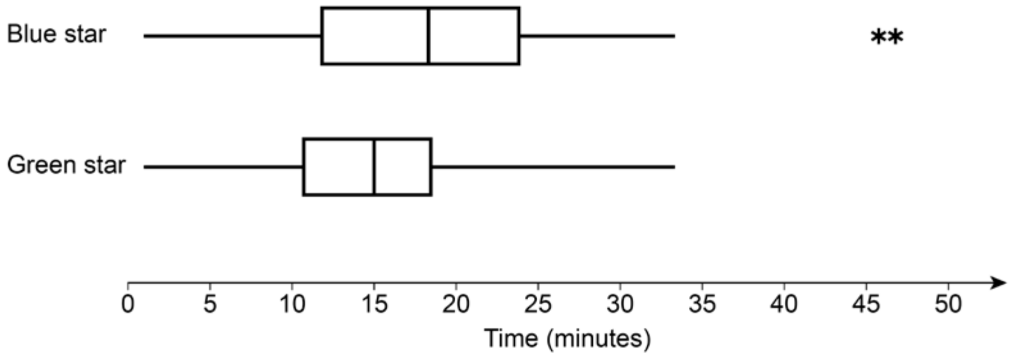
Students should be able to:

- interpret box and whisker plots (boxplots), cumulative frequency curves and histograms

  Note: students will **not** be expected to construct these diagrams.
- Comment on the skewness of a distribution shown in a boxplot. A distribution is positively skewed if Q3 – Q2 > Q2 – Q1 and negatively skewed if Q3 – Q2 < Q2 – Q1
- use diagrams to find probabilities of given events.
  - Note: when studying this topic, students can use data from the large data set and process it using software such as GeoGebra (geogebra.org)
- interpret unfamiliar graphs or representations of data.

Examples

1 Rehana wishes to catch a train from her local station to the city centre. She will take a taxi to the station. There are two local taxi companies; Blue Star and Green Star. In the past Rehana has telephoned both companies. The times, in minutes, that she has to wait between telephoning and the arrival of a taxi are summarised in the boxplot.



(a) Compare briefly the waiting times for the two taxi companies.

(b) Giving a reason for your choice, advise Rehana on which company to telephone if, in order to catch the next train, she needs the taxi to arrive within:

(i) 15 minutes.

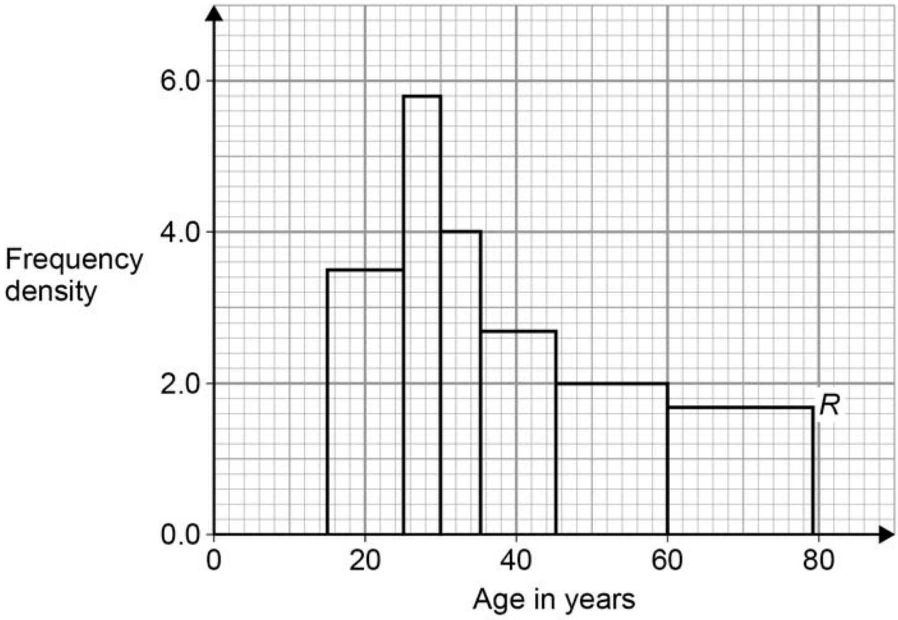
(ii) 25 minutes.

2 Sally's Safaris is a holiday company which organises adventurous holidays. The ages, in years, of the customers who booked holidays in the year 2002 are summarised in the table below.

Age	Frequency
15-24	35
25-29	29
30-34	20
35-44	27
45-59	30
60-79	28

Sally drew the histogram below to illustrate the data. Unfortunately, both coordinates of the point marked *R* have been plotted incorrectly.

State the correct coordinates of the point *R.*



L2

Interpret scatter diagrams and regression lines for bivariate data, including recognition of scatter diagrams which include distinct sections of the population (calculations involving regression lines are excluded).

Understand informal interpretation of correlation.

Understand that correlation does not imply causation.

Assessed at AS and A-level

Teaching guidance

Students should be able to:

- interpret a scatter diagram, to include visual recognition of outliers (as stated in section L4)
- recognise and name positive, negative or no correlation as types of correlation
- understand that, in this qualification, we consider only linear correlation, but that other types are possible. (Link here to F6, where exponential and power laws are considered.)
- recognise and name strong, moderate or weak correlation as strengths of correlation in cases where comparison is possible
- understand that correlation does not imply causality
- state and use the fact that $-1 \le r \le 1$
- interpret, in context, correlation by considering a scatter diagram or a given value of r
- appreciate that interpretation of scatter diagrams by eye is not generally reliable
- interpret the gradient and intercept of a regression line in context.

Note: students will **not** be required to calculate a product moment correlation coefficient. Students will **not** be required to calculate the equation of a regression line or to make predictions using calculations based on the equation of a regression line.

Students will not be expected to plot scatter diagrams, but should be encouraged to use software to plot data from the LDS on scatter diagrams.

Examples

1 Dr Hanna has a special clinic for her older patients. She asked a medical student, Lenny, to select a random sample of 25 of her male patients, aged between 55 and 65 years, and from their clinical records, to list their heights, weights and waist measurements.

Lenny was then asked to calculate three values of the product moment correlation coefficient based upon his collected data. His results were:

(a) 0.365 between height and waist measurement

(b) 1.16 between height and weight

(c) –0.583 between weight and waist measurement.

For each of Lenny's three calculated values, state, with a reason, whether the value is definitely correct, probably correct, probably incorrect or definitely incorrect.

2 Louisa is investigating the link between NOX emissions and engine size for cars in the Large Data Set.

She calculates the regression line to be $y = 0.000005x + 0.012$,

where x = engine size and y = NOX emissions

(a) Interpret the value 0.012 in the regression equation in context, commenting on the validity of this interpretation.

(b) Interpret the value 0.000005 in the regression equation in context.

L3

Interpret measures of central tendency and variation, extending to standard deviation.

Be able to calculate standard deviation, including from summary statistics.

Assessed at AS and A-level

Teaching guidance

Students should be able to:

- find the mean, median, mode, range, quartiles and interquartile range from data given in graphical or tabular form
- interpret values of the mean, median and mode and recognise these as measures of central tendency
- calculate standard deviation (or variance) using a calculator or from summary statistics of the form $\sum x$, $\sum x^2$ or $\sum(x-\bar{x})^2$
- recognise the standard deviation, variance, range and interquartile range as measures of variation.

Notes

- Students are expected to use a calculator's statistical functions to find **all** statistics for a set of data presented as a list or in a frequency table, including estimating for grouped data.
- Linear interpolation of median and quartiles for grouped data is not required.
- Whilst an informal understanding that there are two different values for standard deviation given on a calculator is useful, this specification does not formally address the estimation of population parameters by sample statistics. Thus, the formula we will use for standard deviation is $\sqrt{\dfrac{\sum x^2}{n}-\bar{x}^2}$ (which is given in the formulae book), but students will not be penalised for using the unbiased estimator of the population standard deviation.
- Use of any particular symbol (s or σ) for standard deviation in statistics will be avoided in exam questions, because of the potential for confusion. Students will need to recognise the correct value of standard deviation on a calculator.
- For small data sets, the positions of the median and quartiles are usually given by $\dfrac{n+1}{4}$, $\dfrac{n+1}{2}$, $\dfrac{3(n+1)}{4}$ and it will often be convenient to ensure $n + 1$ is a multiple of 4
- However, the quartiles are more relevant to large sets of data and here it is usually more convenient to replace $n + 1$ by n
- Whenever possible, students should use a calculator to determine quartiles.
- Understand that values of statistics are estimates of the corresponding population parameters.

Examples

1 The runs scored by a cricketer in 11 innings during the 2006 season were as follows.

47 63 0 28 40 51 a 77 0 13 35

The exact value of a was unknown but it was greater than 100

(a) Calculate the median and the interquartile range of these 11 values.

(b) Give a reason why, for these 11 values:

(i) the mode is **not** an appropriate measure of central tendency.

(ii) the range is **not** an appropriate measure of variation.

2 Pat conducted an experiment in which she measured the weight of fat, x grams, in each of a random sample of 10 digestive biscuits, with the following results:

$$\sum x = 31.9 \text{ and } \sum \left(x - \bar{x}\right)^2 = 1.849$$

Use this information to calculate estimates of the mean and standard deviation of the weight of fat in digestive biscuits.

3 The times, in seconds, taken by 20 people to solve a simple numerical puzzle were

17 19 22 26 28 31 34 36 38 39

41 42 43 47 50 51 53 55 57 58

(a) Calculate the mean and the standard deviation of these times.

(b) In fact, 23 people solved the puzzle. However, 3 of them failed to solve it within the allotted time of 60 seconds.

Calculate the median and the interquartile range of the times taken by all 23 people.

(c) For the times taken by all 23 people, explain why:

(i) the mode is **not** an appropriate measure of central tendency

(ii) the range is **not** an appropriate measure of variation

Note: a question in such a context may ask students to find values of possible measures of central tendency and variation for a given set of data, and to critically assess which of these best represents the data.

L4

Recognise and interpret possible outliers in data sets and statistical diagrams.

Select or critique data presentation techniques in the context of a statistical problem.

Be able to clean data, including dealing with missing data, errors and outliers.

Assessed at AS and A-level

Teaching guidance

Students should be able to:

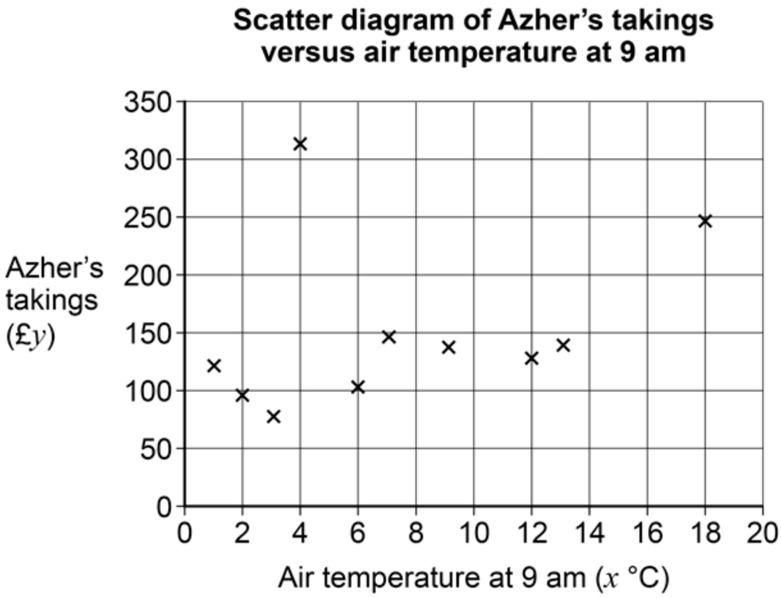
- identify outliers either from a given rule or from observation of a given diagram
- comment on the likely effect of removing the outlier
- identify clear errors in data and comment on or suggest subsequent actions needed
- select which of the representations in sections L1 and L2 is appropriate for representing given data sets
- criticise, in context, a given representation.

Examples

1 Each Monday, Azher has a stall at a town's outdoor market. The table below shows, for each of a random sample of 10 Mondays during 2003, the air temperature, x°C, at 9 am and Azher's takings, £y.

Monday	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
x	2	6	9	18	1	3	7	12	13	4
y	97	103	136	245	121	78	145	128	141	312

(a) A scatter diagram of these data is shown below.



Give **two** distinct comments, in context, on what this diagram reveals.

(b) One of the Mondays is found to be Easter Monday, the busiest Monday market of the year.

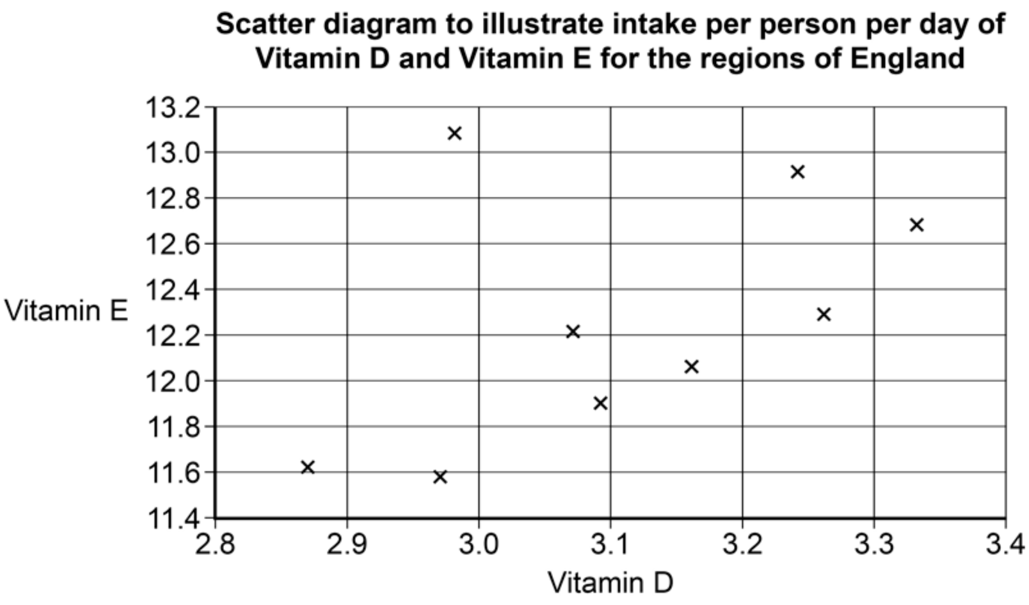
Identify which Monday this is likely to be.

Note: in this question, students are left to decide what the salient features are that are being revealed on the scatter diagram. An ideal response would mention that there is a possible outlier (or that there are two possible outliers).

2 The following table gives information on the average intake, per person per day, of Vitamin D, μg and Vitamin E, mg, for nine regions in England.

Region	**Vitamin D**	**Vitamin E**
North East	2.87	11.63
North West	3.09	11.92
Yorks and Humber	2.97	11.59
East Midlands	3.24	12.95
West Midlands	3.07	12.24
East	3.33	12.72
London	2.98	13.12
South East	3.16	12.09
South West	3.26	12.32

The scatter diagram illustrates this data.



2 (a) Give two distinct comments on what the scatter diagram reveals.

(b) The data for the London region is removed from the table and the data point for London is removed from the scatter diagram.

State what effect this would have on the correlation between average intake of Vitamin D and Vitamin E.

Circle the correct answer.

Correlation would be weaker and positive	Correlation would stay the same	Correlation would be negative	Correlation would be stronger and positive

3 The table below shows data relating to the marital status of the adult populations of England and Wales during the period 1971 to 2006.

		Males					**Females**				
Mid-year	**Total population**	Single	Married	Divorced	Widowed	Total	Single	Married	Divorced	Widowed	Total
1971	36818	4173	12522	187	682	17563	3583	12566	296	2810	19255
1976	37486	4369	12511	376	686	17941	3597	12538		2877	19545
1981	38724	5013	12238	611	698	18559	4114	12284	828	2939	20165
1986	39837	5625	11867	917	695	19103	4617	12000	1165	2953	20734
1991	40501	5891	11636	1187	727	19441	4817	11833	1459	2951	21060
1996	40827	6225	11310	1346	733	19614	5168	11433	1730	2881	21212
2001	41865	6894	11090	1482	733	20198	5798	11150	1975	2745	21667
2006	43494	7833	10881	1696	716	21126	6683	10893	2244	2548	22367

Source: *Population trends*, office for National Statistics, 2009

(a) How many single males were there in England and Wales in 1981?

(b) The number of divorced females for 1976 has been omitted. Calculate the number which should be inserted in that space.

(c) The total population in the table for 1996 is **not** the sum of the total number of males and the total number of females.

Assuming that the figures are correct, explain how this has happened.

M Probability

M1 Understand and use mutually exclusive and independent events when calculating probabilities. Link to discrete and continuous distributions.

Assessed at AS and A-level

Teaching guidance

Students should be able to:

- find the probability of an event by extracting relevant information from a description of a situation (in context) or from a table of information
- recognise and use set theory notation in the context of probability, eg $\mathrm{P}(A \cup B)$, $\mathrm{P}(A \cap B)$, $\mathrm{P}(A')$
- recognise and define the meaning of mutually exclusive events, i.e. $\mathrm{P}(A \cap B) = 0$
- understand that $A \cup B$ means A or B and that, in probability, “or” is interpreted as an inclusive or, not as an exclusive or
- define the condition for two events to be independent and determine whether two events are independent by finding, and comparing, relevant probabilities, eg $\mathrm{P}(A \cap B) = \mathrm{P}(A) \times \mathrm{P}(B)$ or $\mathrm{P}(A) = \mathrm{P}(A \mid B)$, when the events A and B are independent (not required at AS).

Examples

1 Xavier, Yuri and Zara attend a sports centre for their judo club’s practice sessions. The probabilities of them arriving late are, independently, 0.3, 0.4 and 0.2 respectively.

(a) Calculate the probability that for a particular practice session:

(i) all three arrive late.

(ii) none of the three arrives late.

(iii) only Zara arrives late.

(b) Zara’s friend Wei also attends the club’s practice sessions. The probability that Wei arrives late is 0.9 when Zara arrives late, and is 0.25 when Zara does not arrive late.

Calculate the probability that for a particular practice session:

(i) both Zara and Wei arrive late.

(ii) either Zara or Wei, but not both, arrives late.

2 A school employs 75 teachers. The following table summarises their length of service at the school, classified by gender.

	Less than 3 years	3 years to 8 years	More than 8 years
Female	12	20	13
Male	8	15	7

(a) State, giving a reason, whether or not the event of selecting a female teacher is independent of the event of selecting a teacher with less than 3 years' service.

(b) Define an event which is mutually exclusive to the event of selecting a female teacher.

(c) Three teachers are selected at random without replacement, find the probability that all three are:

(i) females with less than three years' service.

(ii) of the same gender.

3 A housing estate consists of 320 houses. The numbers of children living in these houses are shown in the table.

	Number of children				
	None	**One**	**Two**	**At least three**	**Total**
Detached house	24	32	41	23	120
Semi-detached house	40	37	88	35	200
Total	64	69	129	58	320

A house on the state is selected at random.

D denotes the event 'the house is detached'.

R denotes the event 'no children live in the house'.

S denotes the event 'one child lives in the house'.

T denotes the event 'two children live in the house'.

(a) Find $\mathrm{P}(D \cap R)$

(b) (i) Name two of the events D, R, S and T that are mutually exclusive.

(ii) Determine whether the events D and R are independent.

Justify your answer.

(c) Define, in the context of this question, the event $D' \cup T$

M2	Understand and use conditional probability, including the use of tree diagrams, Venn diagrams, two-way tables. Understand and use the conditional probability formula $\mathrm{P}(A\mid B)=\dfrac{\mathrm{P}(A\cap B)}{\mathrm{P}(B)}$

Only assessed at A-level

Teaching guidance

Students should be able to:

- recognise that $\mathrm{P}(A|B)$ is the probability that event A will happen, given that event B has already happened
- find the probability $\mathrm{P}(A|B)$ either by inspection or by using the probability formula $\mathrm{P}(A\cap B)=\mathrm{P}(A)\times\mathrm{P}(B\mid A)=\mathrm{P}(B)\times\mathrm{P}(A\mid B)$ in a given context, which may include the use of tree diagrams, Venn diagrams and two-way tables.

Examples

1 The British and Irish Lions 2005 rugby squad contained 50 players. The nationalities and playing positions of these players are shown in the table.

		Nationality			
		English	**Welsh**	**Scottish**	**Irish**
Playing position	**Forward**	14	5	2	6
	Back	8	7	2	6

A player was selected at random from the squad. Calculate the probability that the player was:

(a) English.

(b) Irish, given that the player was a back.

(c) a forward, given that the player was not Scottish.

2 The age and the blood pressure status for each adult male in a randomly selected sample are summarised in the following table.

	Age range		
Blood pressure status	**25-34**	**35-54**	**55-74**
Normal (untreated)	52	47	28
High (untreated)	4	7	18
High (treated)	1	4	13

One of the adult males is selected at random.

H is the event 'the male has high blood pressure'.

R is the event 'the male selected is aged 55-74'.

S is the event 'the male selected is aged 25-34'.

T is the event 'the male selected is being treated'.

Find:

(a) $\mathrm{P}(H \cap S)$

(b) $\mathrm{P}(S \mid T)$

(c) $\mathrm{P}(R \mid H')$

3 Fred and his daughter, Delia, support their town's rugby team. The probability that Fred watches a game is 0.8 the probability that Delia watches a game is 0.9 when her father watches the game, and is 0.4 when her father does not watch the game.

(a) Calculate the probability that:

(i) both Fred and Delia watch a particular game.

(ii) neither Fred nor Delia watches a particular game.

(b) Molly supports the same rugby team as Fred and Delia. The probability that Molly watches a game is 0.7, and is independent of whether or not Fred or Delia watches the game.

Calculate the probability that:

(i) all three supporters watch a particular game.

(ii) exactly two of the three supporters watch a particular game.

4 On a particular day, a sample of shoppers at a large supermarket were asked which, if any, of three categories of food: meat (M), dairy (D) and fruit and vegetables (F) they intended to buy on their visit that day.

Of these shoppers:

124 answered 'M'

60 answered 'D'

55 answered 'F'

45 answered 'M and D'

12 answered 'D and F'

10 answered 'M and F'

8 answered 'M, D and F'

and 6 answered 'none of the categories'.

(a) Draw a fully labelled Venn diagram to illustrate this information.

(b) State the total number of these shoppers intending to buy only one of the three categories of food.

(c) Find $\mathrm{P}(M \cap D \mid F')$

M3

Modelling with probability, including critiquing assumptions made and the likely effect of more realistic assumptions.

Only assessed at A-level

Teaching guidance

Students should be able to:

- assess and determine whether a stated probability model is appropriate in a given context
- consider whether or not assumptions being made in order to use a given probability model are likely to be valid and the likely effect on results when more realistic assumptions are made.

Examples

1 A gas supplier maintains a team of engineers who are available to deal with leaks reported by customers. Most reported leaks can be dealt with quickly but some require a long time. The time (excluding travelling time), X, taken to deal with reported leaks is found to have a mean of 65 minutes and a standard deviation of 60 minutes.

A statistician consulted by the gas supplier stated that, as the times had a mean of 65 minutes and a standard deviation of 60 minutes, the normal distribution would not provide an adequate model.

Explain the reason for the statistician's statement.

2 During June 2011, the volume, X litres, of unleaded petrol purchased per visit at a supermarket's filling station by private car customers could be modelled by a normal distribution with a mean of 32 and a standard deviation of 10

(a) Determine:

(i) $P(X < 40)$

(ii) $P(X > 25)$

(iii) $P(25 < X < 40)$

(b) Given that during June 2011 unleaded petrol cost £1.34 per litre, calculate the probability that the unleaded petrol bill for a visit during June 2011 by a private car customer exceeded £65.

(c) Give **two** reasons, in context, why the model $N(32,10^2)$ is unlikely to be valid for a visit by **any** customer purchasing fuel at this filling station during June 2011.

3 The proportion of passengers who use senior citizen bus passes to travel into a particular town on park and ride buses between 9.30 am and 11.30 am on weekdays is 0.45

It is proposed that, when there are n passengers on a bus, a suitable model for the number of passengers using senior citizen bus passes is the distribution B(n, 0.45).

(a) Assuming that this model applies to the 10.30 am weekday 'Park and Ride' bus into the town:

(i) calculate the probability that, when there are **16** passengers, exactly three of them are using senior citizen bus passes.

(ii) determine the probability that, when there are **25** passengers, fewer than 10 of them are using senior citizen bus passes.

(iii) determine the probability that, when there are **40** passengers, at least 15 but at most 20 of them are using senior citizen bus passes.

(iv) calculate the mean and the variance for the number of passengers using senior citizen bus passes when there are **50** passengers.

(b) (i) Give a reason why the proposed model may not be suitable.

(ii) Give a **different** reason why the proposed model would not be suitable for the number of passengers using senior citizen bus passes to travel into the town on the **7.15 am** weekday Park and Ride bus.

N Statistical distributions

N1

Understand and use simple, discrete probability distributions (calculation of mean and variance of discrete random variables is excluded), including the binomial distribution, as a model; calculate probabilities using the binomial distribution.

Assessed at AS and A-level

Teaching guidance

Students should:

- recognise when a situation may be modelled by a discrete random variable
- know and be able to use the fact that the sum of the probabilities of all possible outcomes of an event is 1
- understand a discrete probability distribution defined in a table or by a function
- be able to find the probability of a defined event in a given context
- recognise and be able to use $\mathrm{B}(n, p)$ as the notation for a binomial distribution with n independent trials where p is the probability of 'success' at any trial
- be able to state the conditions necessary for a binomial distribution and assess whether they are likely to be valid in a given situation
- be able to find the probability of an exact number of successes in a binomial distribution using the formula (given in the formulae book) or on a calculator
- be able to find cumulative probabilities in a binomial distribution
- calculate the mean, variance and standard deviation of a binomial distribution using the standard formulae given in the formulae book
- be able to use $\mathrm{P}(X \geq n) = 1 - \mathrm{P}(X \leq n-1)$ and similar results

Note: when using the binomial distribution, students must use a calculator to find probabilities.

Examples

1 Todd is a dentist. Clients at Todd's surgery pay one of three possible fees: £20 for a check-up only, £50 for a check-up followed by minor treatment, and £210 for a check-up followed by major treatment. Experience shows that the probabilities for those needing treatment are as in the table.

	Fee	**Probability**
Check-up only	£20	
Check-up + minor treatment	£50	0.32
Check-up + major treatment	£210	0.11

(a) Write down the probability for clients needing a check-up only.

(b) Find the probability that a randomly selected patient in Todd's surgery will pay a fee that is less than £100

2 An amateur tennis club purchases tennis balls that have been used previously in professional tournaments.

The probability that each ball fails a standard bounce test is 0.15

The club purchases boxes each containing 10 of these tennis balls. Assume that the 10 balls in any box represent a random sample.

(a) Determine the probability that the number of balls in a box which fail the bounce test is:

(i) at most 2

(ii) at least 2

(iii) more than 1 but fewer than 5

(b) Determine the probability that, in 5 boxes, the total number of balls which fail the bounce test is:

(i) more than 5

(ii) at least 5 but at most 10

3 Each evening Aaron sets his alarm for 7 am. He believes that the probability that he wakes before his alarm rings each morning is 0.4, and is independent from morning to morning.

One week has seven mornings.

(a) Assuming that Aaron's belief is correct, calculate values for the mean and standard deviation of the number of mornings in a week when Aaron wakes before his alarm rings.

(b) During a 50-week period, Aaron records, each week, the number of mornings on which he wakes before his alarm rings. The results are as follows.

Number of mornings	0	1	2	3	4	5	6	7
Frequency	10	8	7	7	5	5	4	4

(i) Calculate the mean and standard deviation of these data.

(ii) State, giving reasons, whether your answers to part (b) (i) support Aaron's belief that the probability that he wakes before his alarm rings each morning is 0.4, and is independent from morning to morning.

Note: (b) (ii) assesses content from section M3 (A-level only) in that it requires students to critique the assumption that $p = 0.4$ and that it is independent from morning to morning.

4 The probability that a seed of the Staghill daisy will grow is 0.6

(a) Find the probability that at least 5 seeds out of 10 Staghill daisy seeds will grow.

(b) What is the least number of seeds that must be planted so that the probability that at least 5 grow is greater than 90%.

Fully justify your answer.

5 The probability that an egg in a box on a supermarket shelf is broken is 0.02

(a) Find the probability that in a box of six eggs, no more than one egg is broken.

The quality control supervisor inspects a sample of ten boxes of six eggs.

(b) Calculate the probability that in at least two of these ten boxes no more than one egg is broken.

6 The discrete random variable X has distribution

X	2	3	4	5	6
$P(X = x)$	0.32	0.11	0.09	k	k

(a) Find the value of k.

(b) Find $P(X > 3)$

7 The probability of winning a prize of £ X in an online game is given by

$$P(X = x) = \frac{x}{15}, \text{ for } x = 1, 2, 3, 4, 5$$

(a) Explain why the probability of winning a prize other than £1, £2, £3, £4 or £5 is zero.

(b) Find the probability of winning more than £2.50

(c) Sophie plays the game twice. What is the probability she wins exactly £8 in total?

N2

Understand and use the Normal distribution as a model; find probabilities using the Normal distribution.

Link to histograms, mean, standard deviation, points of inflection and the binomial distribution.

Only assessed at A-level

Teaching guidance

Students should:

- know that the normal distribution is a possible model for a continuous random variable
- recognise and be able to use the notation $\mathrm{N}(\mu, \sigma^2)$ to denote the normal distribution with population mean μ and population variance σ^2
- know and be able to use the fact that areas under a normal distribution curve correspond to probabilities
- know and be able to use the symmetry of the normal distribution (this may include using the knowledge that the population mean lies at the centre of a normal distribution)
- know and be able to use the property that the central 99.8% of a normal distribution lies within approximately three standard deviations either side of the mean of the distribution, and other similar results
- understand that a z-score is a measure of how many standard deviations (σ) a value is to the right of the population mean and use the formula $z = \dfrac{x - \mu}{\sigma}$ to find a z-score
- be able to find probabilities using the normal distribution
- recognise when the symmetrical shape of a histogram would allow the normal distribution to be used as a model for the distribution and hence deduce approximate values of the mean and standard deviation of the population
- recognise when the symmetry of a binomial distribution may permit the use of a normal distribution as a model and hence deduce approximate values of the mean and standard deviation of the population (The use of continuity correction is not required.) The actual use of the normal approximation is now unnecessary because calculators can work out binomial distributions that would previously have required such an approximation.
- know and be able to use and show that the points of inflection of a normal distribution lie one standard deviation either side of the mean of the distribution. This is an interesting fact, rather than a deep result in probability theory. Students are not expected to recall the equation of the normal pdf, so if showing that the inflection points are where they are, the required equation would be given
- recognise when a probability found using the normal distribution may be used as the value of p in a binomial distribution and subsequent use of binomial probabilities.

Note: students will be expected to find the values of probabilities in a normal distribution directly from their calculator.

Examples

1 The volume of Everwhite toothpaste in a pump-action dispenser may be modelled by a normal distribution with a mean of 106 ml and a standard deviation of 2.5 ml.

Determine the probability that the volume of Everwhite in a randomly selected dispenser is:

(a) less than 110 ml.

(b) more than 100 ml.

(c) between 104 ml and 108 ml.

(d) not exactly 106 ml.

2 Draught excluder for doors and windows is sold in rolls of nominal length 10 metres.

The actual length, X metres, of draught excluder on a roll may be modelled by a normal distribution with mean 10.2 and standard deviation 0.15

(a) Determine:

(i) $P(X < 10.5)$

(ii) $P(10.0 < X < 10.5)$

(b) A customer randomly selects six 10 metre rolls of the draught excluder.

Calculate the probability that all six rolls selected contain more than 10.5 metres of draught excluder.

3 The random variable X is distributed normally with mean 16.4 and $P(X > 18) = 0.23$

(a) Write down $P(X > 16.4)$

(b) Find the standard deviation of X

(c) Find $P(X < 16)$

4 A scientist studying the Eurasian harvest mouse believes that the mass of an adult mouse may be modelled by a Normal distribution.

She captures a sample of 30 adult harvest mice and compares their masses to two reference masses.

10 mice weigh more than 7 grams. 5 mice weigh more than 10 grams.

(a) Calculate the mean and standard deviation of the mass of an adult Eurasian harvest mouse.

(b) Hence estimate the number of mice in the scientist's sample that would be expected to weigh less than 5 grams

AQA

N3

Select an appropriate probability distribution for a context, with appropriate reasoning, including recognising when the binomial or Normal model may not be appropriate.

Only assessed at A-level

Teaching guidance

Students should:

- know and recognise that the binomial distribution may be used to model a discrete random variable
- know, and be able to state, necessary and sufficient conditions for a discrete random variable to follow a binomial distribution and to recognise when one or more of these conditions may be being violated hence implying the binomial distribution is not a valid model
- be able to use the mean of a binomial distribution and the variance as given in the formulae book)
- know and recognise that the normal distribution may be used to model a continuous random variable
- know and be able to use the properties of the normal distribution, as given in section N2, and recognise when these are being violated hence implying the normal distribution is not a valid model.

Examples

1 In each situation below find the probability of the given event and state whether you are using the binomial distribution, the normal distribution or neither of these in each case.

(a) Past experience suggests that the weight of the contents jars of jam filled by a machine has a mean of 460.2 g and a standard deviation of 2.4 g.

Find the probability that the weight of the contents of a randomly selected jar of jam, filled by this machine, is at least 454 g and at most 464 g.

(b) A sports club has 50 members. Of these members 30 are adults. 15 are juniors and 5 are social members.

Three members are selected at random.

Find the probability that all three are junior members.

(c) A research paper suggests that 10% of the population of the UK is left-handed.

Find the probability that a random sample of 20 people, chosen from this population, contains at least 1 and at most 4 people who are left-handed.

2 A hotel offers guests a choice of a full English breakfast, a continental breakfast or no breakfast. The probabilities of these choices being made are 0.45, 0.25 and 0.30 respectively. It may be assumed that the choice of breakfast is independent from guest to guest.

(a) On a particular morning there are 16 guests. Calculate the probability that exactly 5 of these guests require a full English breakfast.

(b) On another morning, there are 50 guests, determine the probability that:

(i) at most 12 of these guests require a continental breakfast;

(ii) more than 10 but fewer than 20 of these guests require no breakfast.

(c) When there are 40 guests, calculate the mean and the standard deviation for the number of guests requiring breakfast.

3 A supermarket sells multipacks of four 120-gram pots of yoghurt.

The manager is concerned about recent customer complaints about damaged pots and in a survey of yoghurts in the supermarket she has found that 10 yoghurt pots were damaged out of 400 that were checked.

The manager models the number of damaged pots in a multipack by the random variable $X \sim \mathrm{B}(4, 0.025)$

(a) Use the manager's model to calculate the probability that there will be at least one damaged pot in a randomly selected multipack.

(b) Comment on the validity of the manager's use of the probability of 0.025 in her binomial model.

0 Statistical hypothesis testing

01

Understand and apply the language of statistical hypothesis testing, developed through a binomial model: null hypothesis, alternative hypothesis, significance level, test statistic, 1-tail test, 2-tail test, critical value, critical region, acceptance region, p-value; extend to correlation coefficients as measures of how close data points lie to a straight line and be able to interpret a given correlation coefficient using a given p-value or critical value (calculation of correlation coefficients is excluded).

Assessed at AS and A-level

Teaching guidance

Students should:

- recognise whether a given context requires the use of a 1-tail or 2-tail test and understand the difference between them
- be able to state appropriate null and alternative hypotheses to test a population proportion in a given context and know that the null hypothesis always contains the equality sign
- understand that the significance level of a test is the probability of rejecting a correct null hypothesis in error
- be able to find the test statistic, which is the observed number of outcomes of the event
- be able to find the critical region for a 1-tail test, or the critical regions for a 2-tail test, supporting the choice of values in such regions with appropriate binomial probabilities
- know that the critical region consists of the critical values for the test and that if the test statistic lies in the critical region then the null hypothesis is rejected
- know that the acceptance region is the range of possible values, that the discrete random variable can take, that do not lie in the critical region and that if the test statistic lies in the acceptance region that this will lead to the acceptance of the null hypothesis
- be able to use the given p-value corresponding to the test statistic or the given critical value(s), for the relevant significance level of the test, to decide whether to accept or reject the null hypothesis; understand that the p-value should be compared to a binomial distribution critical region with probability equal to or less than the significance level
- know that the precise definition of a p-value in a 2-tailed test varies. It can be defined as the

probability

calculated from the test statistic or twice that value. In order to circumvent this difficulty, questions will not be asked in which students are required to state the p-value for such a test

- be able to interpret a conclusion in context.

Notes

- The conclusion of a hypothesis test is an inference based on evidence and thus students must indicate that there is no certainty in their conclusions. Using the phrase "sufficient evidence to suggest" (qualified with a "not" as applicable) would be a good standard to adopt. There is nothing to be gained by trying to write a conclusion creatively. The final concluding statement of a hypothesis test should always relate back to the context.
- In cases where the null hypothesis is not rejected we would allow an inference of "Do not reject H0" or "Accept H0." Statistical purists will prefer the former.

Examples

1 A test statistic has a binomial distribution, B(25, 0.15)

Given that

$H_0 : p = 0.15 \qquad H_1 : p < 0.15$

(a) Find the critical region for the test statistic when the level of significance for the test is 5%.

Note: the critical region is the set of values of the test statistic that leads to the rejection of the null hypothesis, when the null hypothesis is assumed true. It is the area of the sampling distribution of a statistic that will lead to the rejection of the hypothesis tested when that hypothesis is true.

(b) Find the probability of rejecting the null hypothesis, even though it is actually true, when the critical region found in part (a) is used.

2 Previous experience suggests that 15% of the workers in a factory wear glasses.

Arwen selects a random sample of 20 workers from the factory and finds that 6 of them are wearing glasses. She suspects that the proportion of workers in the factory wearing glasses may have changed.

She uses the hypotheses

$H_0 : p = 0.15 \qquad H_1 : p > 0.15$

and decides to use a 5% significance level for her test.

Her test statistic of '6 workers are wearing glasses' gives her a p-value of 0.06731

Complete Arwen's test.

Only assessed at A-level

Teaching guidance

Students should be able to:

- carry out a hypothesis test for a product moment correlation coefficient.

Notes

- There are no tables of correlation coefficient critical values and students will be given an appropriate critical value or probability to compare with. These values may be given in an extract from a table of critical values.

Examples

1 The correlation coefficient between the mean daily temperature and the sales of socks in a large supermarket for data collected over a period of ten days is found to be –0.57

The critical value of the correlation coefficient at the 5% level is 0.549

Carry out a hypothesis test to determine if there is evidence that as the mean daily temperature increases the sales of socks decrease.

2 Felix is testing the correlation between the daily sunshine hours and the mean growth of the stems of privet.

He collects data on 20 days and then finds the product moment correlation coefficient between daily sunshine hours and mean growth, obtaining a value of 0.334

He writes down the following hypotheses:

H_0: $r = 0$

H_1: $r = 0.334$

Felix has made two different errors in formulating these hypotheses.

Explain the errors Felix has made.

02

Conduct a statistical hypothesis test for the proportion in the binomial distribution and interpret the results in context.

Understand that a sample is being used to make an inference about the population and appreciate that the significance level is the probability of incorrectly rejecting the null hypothesis.

Assessed at AS and A-level

Teaching guidance

Students should be able to:

- state appropriate null and alternative hypotheses to test the value of a population proportion in a given context
- find the test statistic as being the observed number of outcomes of the event
- either find the critical region(s) for the test (supporting the choice of critical region with appropriate binomial probabilities) or find the p-value corresponding to the test statistic; understand that the p-value should be compared to a binomial critical region with probability equal to or less than the significance level
- either compare the test statistic with the critical region or compare the p-value with the significance level of the test
- state the conclusion in context
- appreciate that the results obtained from the sample are being used to make an inference as to what is happening in the parent population and that the conclusion reached may be incorrect.

Examples

1 National records show that 35% of train passengers buy their tickets in advance. A random sample of 25 passengers using a particular railway station is selected, and it is found that 13 of them bought their tickets in advance.

Investigate, at the 10% level of significance, whether the data support the view that the percentage of passengers from this station who buy their tickets in advance is different from the national figure of 35%.

Note: the actual level of significance of this data is less than 10%.

2 James is a guitarist in a rock band which is about to start a 14-night tour. James usually uses Britepick guitar strings, which he changes before each performance. The thinnest string on a guitar, the top-E string, is the one most likely to break and, for James, the probability that this happens during a 1-hour performance is 0.02

(a) James is thinking of using Pluckwell strings rather than Britepick strings in the future and has bought some Pluckwell top-E strings to use each night of the 14-night tour. He finds that he breaks a top-E string during the band's 1-hour performance on two of these 14 nights.

Investigate, at the 5% level of significance, whether Pluckwell top-E strings are more likely to break than Britepick top-E strings.

(b) The band's manager, Noddy, suggests that the conclusion that James reached in part (a) is always going to be the same for these two makes of string.

Comment on the validity of Noddy's suggestion.

Note: in part (b) students need to decide how best to approach this part of the question. Solutions may include that the conclusion is only based on the result arrived at by use of a small sample of strings and that this may not be true for all strings produced by the company – the breaking of strings may be influenced by temperature/humidity/striking the strings harder (or softer) on the tour. They may also conclude that with modern quality control the strings produced are likely to be extremely consistent and that although it is likely similar results will be achieved with similar testing conditions we can't say that the result will always be the same since there will be variability within output.

03

Conduct a statistical hypothesis test for the mean of a Normal distribution with known, given or assumed variance and interpret the results in context.

Only assessed at A-level

Teaching guidance

Students should be able to:

- state appropriate null and alternative hypotheses to test the value of a population mean in a given context
- find the test statistic using the formula $\dfrac{\overline{X}-\mu}{\left(\dfrac{\sigma}{\sqrt{n}}\right)}$
- either find the critical value(s) for the test or find the p-value corresponding to the test statistic
- either compare the test statistic with the critical value(s) or compare the p-value with the significance level of the test
- decide whether to accept or reject the null hypothesis
- state the conclusion in the context of the question
- appreciate that the results obtained from the sample are being used to make an inference as to what is happening in the parent population and that the conclusion reached may be incorrect.

Notes

- The use of the term "standard error" is not required.
- Students could be asked to calculate probabilities related to the sample mean.

Examples

1 As a special promotion, a supermarket offers cartons of orange juice containing '25% extra' with no price increase.

A random sample of cartons of orange juice was checked. The percentages by which the contents exceeded the nominal quantity were recorded, with the following results:

23.3 27.5 25.7 20.9 24.3 22.6 21.5 22.1

Examine whether the mean percentage by which the contents exceed the nominal quantity is less than 25. Use the 5% significance level. Assume that the data are from a normal distribution with standard deviation 2.3

2 A machine fills paper bags with flour. Before maintenance on the machine, the weight of the flour in a bag could be modelled by a normal distribution with mean 1005 g and standard deviation 2.1 grams. Following this maintenance, the flour in each of a random sample of 8 bags was weighed. The weights, in grams, were as follows:

1006.1 1004.9 1005.8 1007.9 1004.7 1006.3 1007.4 1007.2

(a) Carry out a test at the 10% significance level, to decide whether the mean weight of flour in a bag filled by the machine had **changed**. Assume that the distribution of weights was still normal with standard deviation 2.1 grams.

(b) Calculate the probability that the mean weight of a sample of 8 bags of flour lies between 1004.5 and 1005.5 grams.

P Quantities and units in mechanics

P1 Understand and use fundamental quantities and units in the SI system: length, time, mass.

Understand and use derived quantities and units: velocity, acceleration, force, weight, moment.

Assessed at AS and A-level

Teaching guidance

Students should:

- know and be able to use the following:

Fundamental quantity	**SI base unit**
length	metre (m)
time	second (s)
mass	kilogram (kg)

- be able to convert between commonly used SI units, for example kilometres and metres or kilograms and tonnes
- know and be able to use the following:

Derived quantity	**SI unit**
velocity	metre per second (m s^{-1})
acceleration	metre per second squared (m s^{-2})
force/weight	newton (N)
moment	newton metre (Nm) [**A-level only**]

- understand that understand that weight is a force, $W = mg$ (N)
- if required, be able to convert non-standard units into standard units, for example kilometres per hour to metres per second
- be familiar with equivalent notations such as m/s^2 and m s^{-2}
- know that trigonometric ratios and the coefficient of friction do not have units
- know that g is acceleration due to gravity (m s^{-2}) and in questions where a value of g is given, the final answer should be given to the same degree of accuracy as the value of g
- know that all other values used in Mechanics questions are treated as exact values.

Example

1 A car of mass 1.2 tonnes is accelerating at 5 m s^{-2}

Calculate the magnitude of the resultant force acting on the car.

Only assessed at A-level

Example

1 A uniform rod, of length 1.2 metres, has a mass of 3 kg. The rod is held in equilibrium by two supports, *A* and *B*, as shown in the diagram.

Calculate the clockwise moment of the weight of the rod about support *A.*

Circle your answer.

45 Nm 3.6 kg/m 1.35g Nm 3g N

Q Kinematics

Q1 Understand and use the language of kinematics: position; displacement; distance travelled; velocity; speed; acceleration.

Assessed at AS and A-level

Teaching guidance

Students should be able to:

- understand positions described relative to a given origin
- understand and describe the position of a particle through a combination of its initial position and a displacement
- demonstrate an understanding of the relationship between the vector quantities displacement and velocity and their associated scalar quantities distance and speed
- understand average speed and average velocity.

Examples

1 A car travels on a straight horizontal race track. The car decelerates uniformly from a speed of 20 m s^{-1} to a speed of 12 m s^{-1} as it travels a distance of 640 metres. The car then accelerates uniformly, travelling a further 1820 metres in 70 seconds.

(a) (i) Find the time that it takes the car to travel the first 640 metres.

(ii) Find the deceleration of the car during the first 640 metres.

(b) (i) Find the acceleration of the car as it travels the further 1820 metres.

(ii) Find the speed of the car when it has completed the further 1820 metres.

2 Two boys, Alfie and Bruce, are running a 100 metres race. After 4 seconds Alfie has run 21.5 metres and Bruce has run 18.3 metres and both boys have reached their maximum speeds. The boys complete the race running at their maximum speeds. Alfie's maximum speed is 5.9 m s^{-1} and Bruce's maximum speed is 6.2 m s^{-1}

Determine:

(a) the total time taken by the winner of the race.

(b) the distance from the finish of the boy in second place when the race is won.

Q2

Understand, use and interpret graphs in kinematics for motion in a straight line: displacement against time and interpretation of gradient; velocity against time and interpretation of gradient and area under the graph.

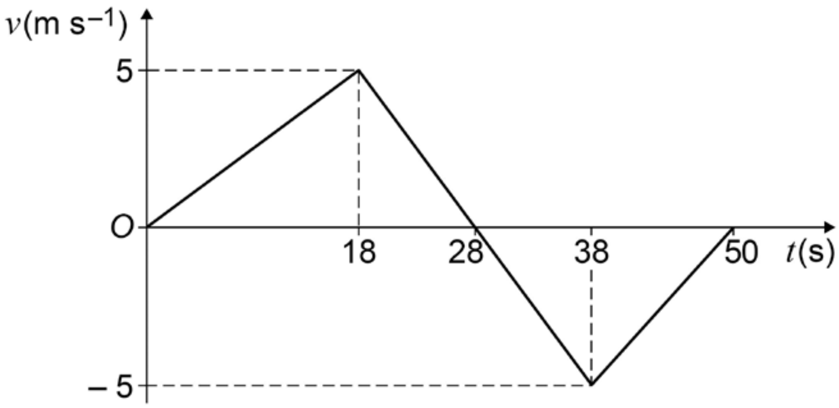
Assessed at AS and A-level

Teaching guidance

Students should be able to:

- use the gradient of a displacement-time graph to find the velocity (or speed)
- use the gradient of a velocity-time graph to find the acceleration and interpret positive and negative gradients
- understand that graphs may include negative velocities
- use the area under a velocity-time graph to find displacement
- sketch either a displacement-time or velocity-time graph for a given scenario.

1 The diagram shows a velocity-time graph for a train as it moves on a straight horizontal track for 50 seconds.



(a) Find the distance that the train moves in the first 28 seconds.

(b) Calculate the total distance moved by the train during the 50 seconds.

(c) Find the displacement of the train from its initial position when it has been moving for 50 seconds.

(d) Find the acceleration of the train in the first 18 seconds of its motion.

2 A van moves from rest on a straight horizontal road.

(a) In a simple model, the first 30 seconds of the motion are represented by three separate stages, each lasting 10 seconds and each with a constant acceleration.

During the first stage, the van accelerates from rest to a velocity of 4 m s^{-1}

During the second stage, the van accelerates from 4 m s^{-1} to 12 m s^{-1}

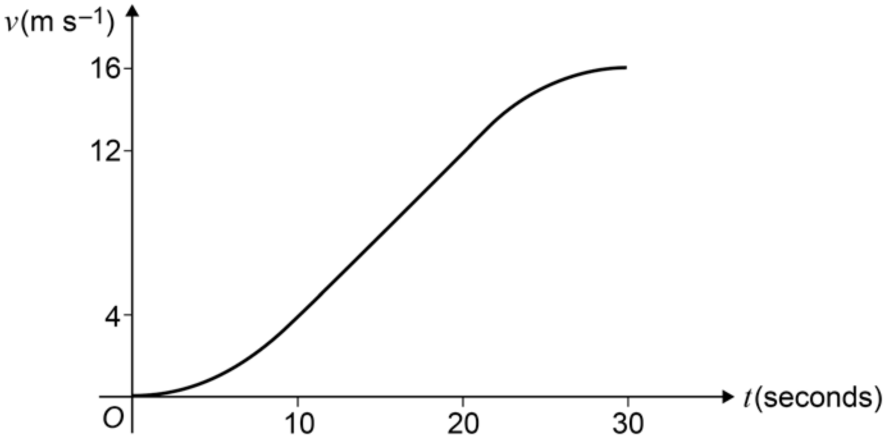
During the third stage, the van accelerates from 12 m s^{-1} to 16 m s^{-1}

(i) Sketch a velocity-time graph to represent the motion of the van during the first 30 seconds of its motion.

(ii) Find the total distance that the van travels during the 30 seconds.

(iii) Find the greatest acceleration of the van during the 30 seconds.

(b) In another model of the 30 seconds of the motion, the acceleration of the van is assumed to vary during the first and third stages of the motion, but to be constant during the second stage, as shown in the velocity-time graph below.

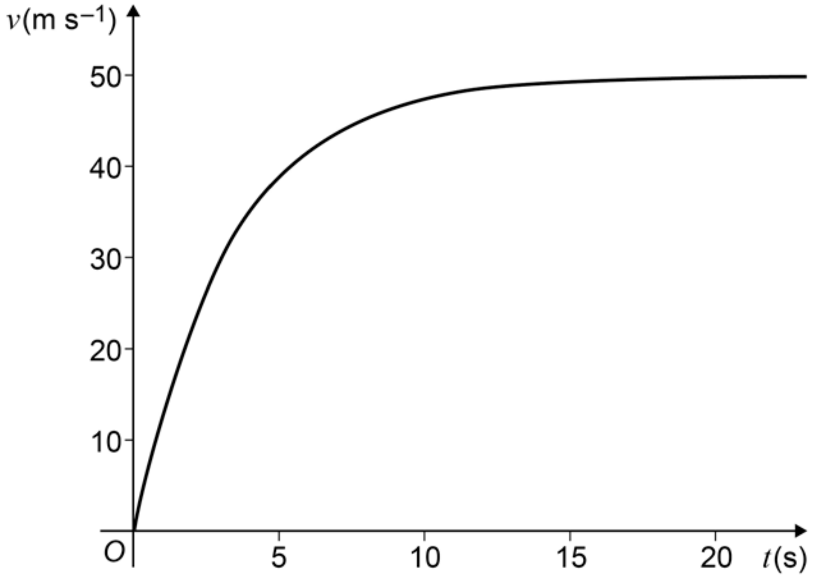


The velocity of the van takes the same values at the beginning and the end of each stage of the motion as in part (a).

(i) State, with a reason, whether the distance travelled by the van during the first 10 seconds of the motion in **this** model is greater or less than the distance travelled during the same interval in the model in part (a).

(ii) Give one reason why **this** model represents the motion of the van more realistically than the model in part (a).

3 The graph shows the velocity of a parachutist t seconds after jumping from a hot-air balloon, before she opens her parachute.



What is the value of t when her acceleration is at a maximum?

Q3

Understand, use and derive the formulae for constant acceleration for motion in a straight line; extend to 2 dimensions using vectors.

Assessed at AS and A-level

Teaching guidance

Students should be able to:

- recall and use the following formulae:

$$v = u + at$$

$$s = \frac{1}{2}(u+v)t$$

$$s = ut + \frac{1}{2}at^2$$

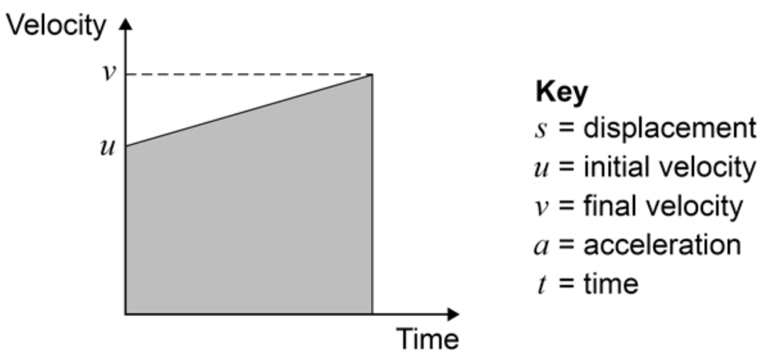
$$s = vt - \frac{1}{2}at^2$$

$$v^2 = u^2 + 2as$$

Note: the less fashionable $s = vt - \frac{1}{2}at^2$ is not essential.

- derive the above formulae starting from given assumptions.

  This may include starting from a graph similar to the one below.



For example, the shaded area gives the displacement, s, and can be found as the area of a trapezium:

$$s = \frac{1}{2}(u+v)t$$

Example

1 Two cameras record the time that it takes a car on a motorway to travel a distance of 2 km. A car passes the first camera travelling at 32 m s^{-1}. The car continues at this speed for 12.5 seconds and then decelerates uniformly until it passes the second camera when its speed has decreased to 18 m s^{-1}

(a) Calculate the distance travelled by the car in the first 12.5 seconds.

(b) Find the time for which the car is decelerating.

(c) Sketch a velocity-time graph for the car on this 2 km stretch of motorway.

(d) Find the total time taken for the car to travel along this 2 km stretch of motorway.

Only assessed at A-level

Teaching guidance

Students should be able to extend linear constant acceleration equations using the notation:

$\mathbf{r}$ = position vector
$\mathbf{r}_0$ = initial position
$\mathbf{u}$ = initial velocity
$\mathbf{v}$ = final velocity
$\mathbf{a}$ = acceleration
t = time

Linear constant acceleration equation	Vector constant acceleration equation
$v = u + at$	$\mathbf{v} = \mathbf{u} + \mathbf{a}t$
$s = \frac{1}{2}(u+v)t$	$\mathbf{r} = \frac{1}{2}(\mathbf{u}+\mathbf{v})t + \mathbf{r}_0$
$s = ut + \frac{1}{2}at^2$	$\mathbf{r} = \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2 + \mathbf{r}_0$
$s = vt - \frac{1}{2}at^2$	$\mathbf{r} = \mathbf{v}t - \frac{1}{2}\mathbf{a}t^2 + \mathbf{r}_0$
$v^2 = u^2 + 2as$	No equivalent

Note: when working with vectors, students can use column vectors $\begin{bmatrix} a \\ b \end{bmatrix}$ or express them in the form $a\mathbf{i} + b\mathbf{j}$.

Example

1 A helicopter is initially hovering above a lighthouse. It then sets off so that its acceleration is $(0.5\mathbf{i} + 0.375\mathbf{j})$ m s^{-2}. The helicopter does not change its height above sea level as it moves. The unit vectors **i** and **j** are directed east and north respectively.

(a) Find the speed of the helicopter 20 seconds after it leaves its position above the lighthouse.

(b) Find the bearing on which the helicopter is travelling, giving your answer correct to the nearest degree.

(c) The helicopter stops accelerating when it is 500 metres from its initial position. Find the time that it takes for the helicopter to travel from its initial position to the point where it stops accelerating.

Q4

Use calculus in kinematics for motion in a straight line: $v=\frac{\mathrm{d}r}{\mathrm{d}t}$, $a=\frac{\mathrm{d}v}{\mathrm{d}t}=\frac{\mathrm{d}^2r}{\mathrm{d}t^2}$, $r=\int v\,\mathrm{d}t$, $v=\int a\,\mathrm{d}t$; extend to 2 dimensions using vectors.

Assessed at AS and A-level

Teaching guidance

Students should:

- know and be able to apply the following to motion in a straight line:

$$v=\frac{\mathrm{d}r}{\mathrm{d}t} \qquad a=\frac{\mathrm{d}v}{\mathrm{d}t}=\frac{\mathrm{d}^2r}{\mathrm{d}t^2}$$

$$r=\int v\,\mathrm{d}t \qquad v=\int a\,\mathrm{d}t$$

- explore the relationship between calculus methods and the constant acceleration equations.

Note: at AS, questions will be limited to integrating or differentiating functions formed from sums and differences of terms such as At^n (with $n\neq-1$ in the case of integration), matching the level of difficulty required for sections G and H at AS.

Examples

1 A particle moves in a straight line and at a time t it has velocity v, where

$$v=3t^2-12t+6$$

(a) (i) Find an expression for the acceleration of the particle at time t

(ii) When $t=2$, show that the acceleration of the particle is 0

(b) When t = 0, the particle is at the origin.

Find an expression for the displacement of the particle from the origin at time t

2 A particle moves in a straight line with constant acceleration, a

Given that the initial velocity of the particle is u, use integration to prove that the displacement of the particle from its initial position is given by

$$s=ut+\frac{1}{2}at^2$$

Only assessed at A-level

Teaching guidance

Students should be able to:

- answer questions that draw on any of the differentiation or integration techniques from the pure mathematics sections G and H, for example:
  - find the velocity, given that the acceleration, a, at time t is given by $a = t^2e^{-t}$
  - find the acceleration given that the displacement, s, at time t is given by $s = e^{\frac{-t}{2}}\sin\left(\frac{nt}{20}\right)$
- recall and use the facts that in two dimensions, if $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$ then:

$$\mathbf{v} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} \text{ and } \mathbf{a} = \frac{d^2x}{dt^2}\mathbf{i} + \frac{d^2y}{dt^2}\mathbf{j}$$

Examples

1 A particle moves in a straight line and at time t it has velocity v, where

$$v = 3t^2 - 2\sin 3t + 6$$

(a) (i) Find an expression for the acceleration of the particle at time t

(ii) When $t = \frac{\pi}{3}$, show that the acceleration of the particle is $2\pi + 6$

(b) When $t = 0$, the particle is at the origin.

Find an expression for the displacement of the particle from the origin at time t

2 A particle moves on a horizontal plane, in which the unit vectors **i** and **j** are directed east and north respectively.

At time t seconds, the position vector of the particle is **r** metres, where

$$\mathbf{r} = \left(2e^{\frac{1}{2}t} - 8t + 5\right)\mathbf{i} + \left(t^2 - 6t\right)\mathbf{j}$$

(a) Find an expression for the velocity of the particle at time t

(b) (i) Find the speed of the particle when $t = 3$

(ii) State the direction in which the particle is travelling when $t = 3$

(c) Find the acceleration of the particle when $t = 3$

(d) The mass of the particle is 7 kg. Find the magnitude of the resultant force on the particle when $t = 3$

Q5 Model motion under gravity in a vertical plane using vectors; projectiles.

Assessed at A-level only

Teaching guidance

Students should be able to:

- use the acceleration due to gravity in the vector form:

  $\mathbf{a} = -g\mathbf{j}$ or $\mathbf{a} = \begin{bmatrix} 0 \\ -g \end{bmatrix}$

  where $\mathbf{j}$ is defined as vertically upwards from the earth's surface.

  Notes: questions could be set in the context of vertical motion or in two dimensions.

  In projectile questions, students can expect a wide range of scenarios.

  When solving equations:

  • calculators may be used to solve quadratic equations. It is recommended that both solutions are always given by students and the appropriate solution selected, with explicit justification for the selection.

  • students may need to use the following trigonometric identities:

  $\sin 2\theta = 2\sin\theta\cos\theta$

  $\sec^2\theta = 1 + \tan^2\theta$

- use the value of 9.8, 9.81 or 10 for g, as directed in the question and give any numerical answer to the same accuracy as the value of g
- use and understand assumptions made when modelling projectiles. For example:

  • projectile is a particle (has no size and does not spin).

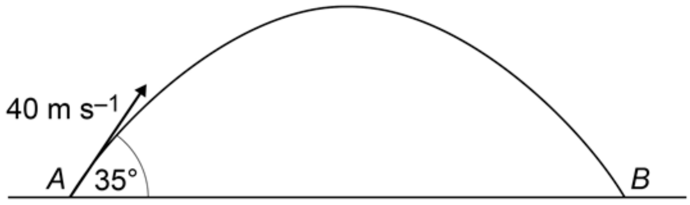
  • projectile does not experience air resistance or wind.
- find the equation of the trajectory of a projectile.

Examples

1 **In this question use g = 9.8 m s^{-2}**

A ball is hit by a bat so that, when it leaves the bat, its velocity is 40 m s^{-1} at an angle of 35° above the horizontal. Assume that the ball is a particle and that its weight is the only force that acts on the ball after it has left the bat.

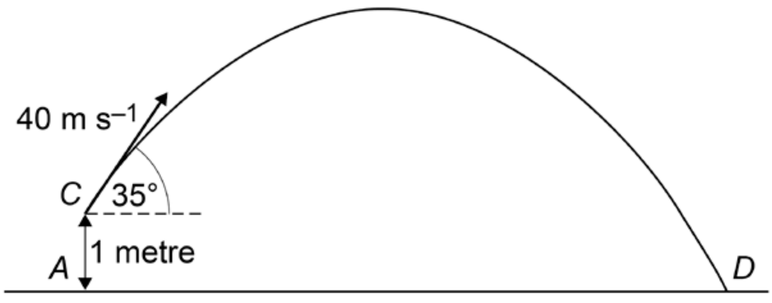
(a) A simple model assumes that the ball is hit from the point *A* and lands for the first time at the point *B*, which is at the same level as *A*, as shown in the diagram.



(i) Show that the time that it takes for the ball to travel from *A* to *B* is 4.7 seconds.

(ii) Find the horizontal distance from *A* to *B*.

(b) A revised model assumes that the ball is hit from the point *C*, which is 1 metre above *A*. the ball lands at the point *D*, which is at the same level as *A*, as shown in the diagram.



Find the time that it takes for the ball to travel from *C* to *D*.

2 **In this question use g = 10 m s^{-2}**

A projectile is launched at ground level, on a horizontal surface, with speed V at an angle θ above the horizontal.

(a) Show that the range of the projectile is $\dfrac{V^2 \sin 2\theta}{g}$ and state any assumptions that you make to obtain this result.

(b) Find an expression for the maximum height of the projectile.

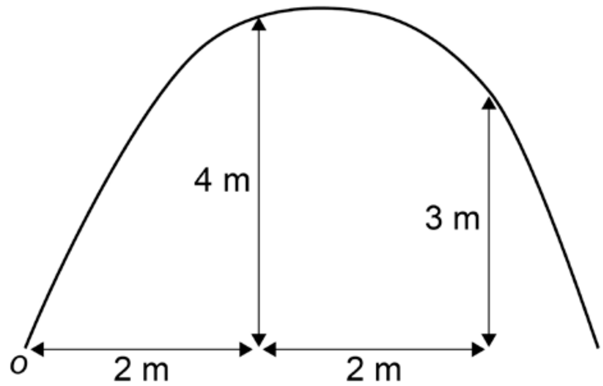
(c) Calculate the range of a projectile launched at a speed of 30 m s^{-1} and at an angle of 40° above the horizontal.

3 **In this question use g = 10 m s^{-2}**

A ball is thrown at an angle α above the horizontal with an initial speed of V m s^{-1}. At time t seconds the horizontal displacement of the ball from its initial position, *O*, is x metres and the vertical displacement is y metres. Assume that the only force acting on the ball after it has been thrown is its weight.

(a) Show that $y = x\tan\alpha - \dfrac{gx^2}{2V^2}(1+\tan^2\alpha)$

The ball is thrown from *O*, so that it passes through two small hoops. The hoops are set at horizontal distances of 2 m and 4 m from *O*. The positions of the hoops are shown in the diagram. Assume that the ball is a particle that passes through the centre of each hoop.



(b) Show that $\tan\alpha = \dfrac{13}{4}$ and find V

(c) If a heavier ball was thrown with the same initial velocity, would it pass through the hoops? Give a reason for your answer.

R Forces and Newton's laws

R1 Understand the concept of a force; understand and use Newton's first law.

Assessed at AS and A-level

Teaching guidance

Students should:

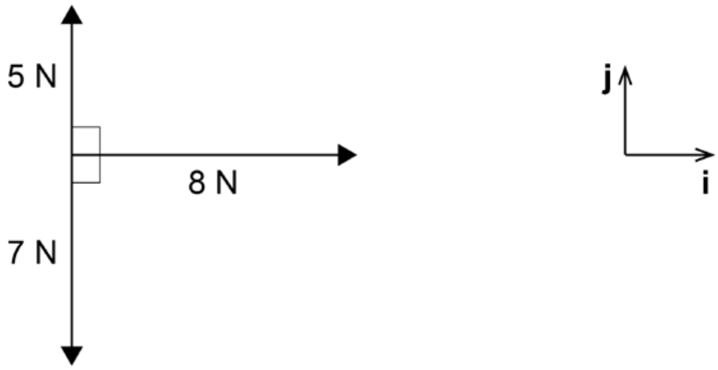
- understand types of force, including:
  - normal reaction force
  - tension in a string or a rod
  - thrust in a rod
  - weight
  - friction.
- know that the resultant force acting on a body is zero if a body is in equilibrium
- be able to find unknown forces acting on bodies that are at rest or moving with constant velocity
- be able to model forces as vectors
- be able to find the resultant of several forces acting at a point.

Notes

- Students may be required to express a resultant force using components of a vector. Students may be required to find the magnitude and direction of a resultant force expressed as a vector.
- Students will not be required to resolve forces at AS. This means that problems at AS will not be set in contexts of inclined planes or strings at an angle, for example. At AS forces will only be perpendicular or parallel to other forces, unless they are given as vectors in component form.

Examples

1 The diagram shows three forces and the perpendicular unit vectors **i** and **j**, which all lie in the same plane.



(a) Express the resultant of the three forces in terms of **i** and **j**

(b) Find the magnitude of the resultant force.

(c) Draw a diagram to show the direction of the resultant force, and find the angle that it makes with the unit vector **i**

2 A car is travelling with a constant velocity of 15 m s^{-1}, in a straight line, on a horizontal road. A driving force of 600 N acts in the direction of motion and a resistance force opposes the motion of the car. Assume that no other horizontal forces act on the car.

What is the magnitude of the resistance force acting on the car?

Circle the correct answer.

600 N 40 N 9000 N 0 N

Only assessed at A-level

Teaching guidance

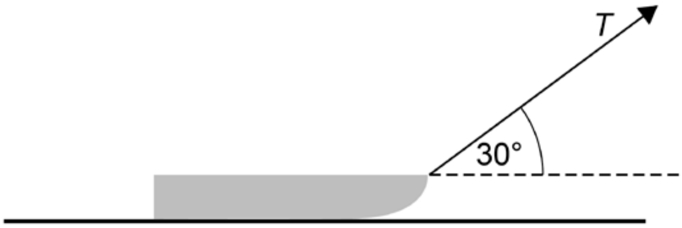
Students should be able to:

- model forces as vectors
- resolve forces, but only in two dimensions.

Example

1 **In this question, use $g = 10$ m s^{-2}**

A child pulls a sledge, of mass 8 kg, along a rough horizontal surface, using a light rope. The coefficient of friction between the sledge and the surface is 0.3. The tension in the rope is *T* newtons. The rope is kept at an angle of 30° to the horizontal, as shown in the diagram.



Model the sledge as a particle.

(a) Draw a diagram to show all the forces acting on the sledge.

(b) Find the magnitude of the normal reaction force acting on the sledge in terms of *T*.

(c) Given that the sledge moves with a constant velocity, find the value of *T*.

R2

Understand and use Newton's second law for motion in a straight line (restricted to forces in two perpendicular directions or simple cases of forces given as 2-D vectors); extend to situations where forces need to be resolved (restricted to 2 dimensions).

Assessed at AS and A-level

Teaching guidance

Students should be able to:

- use $F = ma$ for constant mass and constant force
- understand that objects can be modelled as particles
- comment on the relevance of any modelling assumptions made.

Note: questions may be set that require the use of constant acceleration equations together with Newton's second law.

Examples

1 A lift rises vertically from rest with constant acceleration.

After 4 seconds, it is moving upwards with a velocity of 2 m s^{-1}

It then moves with a constant velocity for 5 seconds.

The lift then slows down uniformly, coming to rest after it has been moving for a total of 12 seconds.

(a) Sketch a velocity-time graph for the motion of the lift.

(b) Calculate the total distance travelled by the lift.

(c) The lift is raised by a single vertical cable. The mass of the lift is 300 kg.

Find the maximum tension in the cable during this motion.

2 Three forces act on a particle. These forces are (9**i** – 3**j**) newtons, (5**i** + 8**j**) newtons and (−7**i** + 3**j**) newtons. The vectors **i** and **j** are perpendicular unit vectors.

(a) Find the resultant of these forces.

(b) Find the magnitude of the resultant force.

(c) Given that the particle has mass 5 kg, find the magnitude of the acceleration of the particle.

(d) Find the angle between the resultant force and the unit vector **i**

Only assessed at A-level

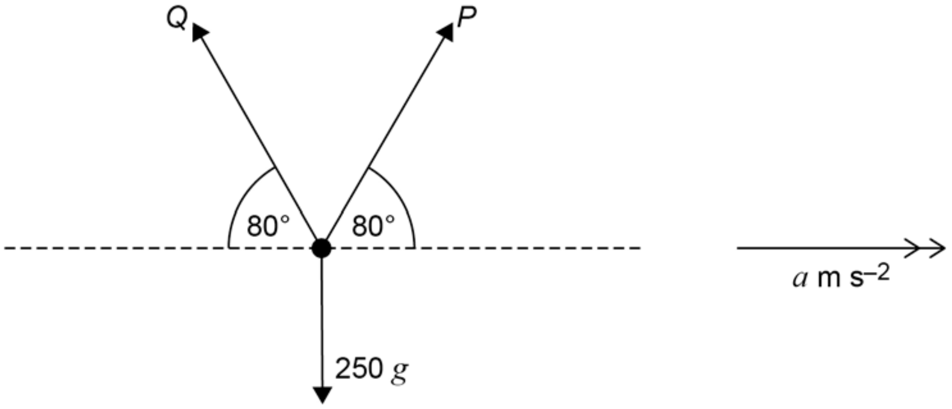
Teaching guidance

Students should be able to answer questions set on an inclined plane or in other contexts that require forces to be resolved.

Examples

1 **In this question, use g = 9.8 m s^{-2}**

Three forces act in a vertical plane on an object of mass 250 kg, as shown in the diagram.



The two forces P newtons and Q newtons each act at 80° to the horizontal. The object accelerates horizontally at a m s^{-2} under the action of these forces.

(a) Show that

$$P = 125\left(\frac{a}{\cos 80^\circ} + \frac{g}{\sin 80^\circ}\right)$$

(b) Find the value of a when Q is zero.

R3

Understand and use weight and motion in a straight line under gravity; gravitational acceleration, g, and its value in SI units to varying degrees of accuracy.

(The inverse square law for gravitation is not required and g may be assumed to be constant, but students should be aware that g is not a universal constant but depends on location).

Assessed at AS and A-level

Teaching guidance

Students should be able to:

- understand the distinction between mass and weight
- state necessary modelling assumptions, and relate these to the model used.

Notes

- In questions where a numerical value for g is needed, students will be clearly told which approximation to use and their answers must then be given to the same accuracy: 10 requires 1 significant figure, 9.8 two significant figures and 9.81 three significant figures.
- In questions involving objects in motion under gravity it will be assumed that:
  - g remains constant
  - objects can be treated as particles
  - resistance forces are negligible.

Examples

1 **In this question, use g = 10 m s^{-2}**

A stone is dropped from a high bridge and falls vertically.

(a) Find the distance that the stone falls during the first 4 seconds of its motion.

(b) Find the speed of the stone after the first 4 seconds of its motion.

(c) State one modelling assumption that you have made about the forces acting on the stone.

2 **In this question, use g = 9.8 m s^{-2}**

On earth, an astronaut wearing her full spacesuit weighs 2100 newtons.

On the moon, wearing the same spacesuit, she weighs 350 newtons.

Calculate the acceleration due to gravity on the moon.

R4

Understand and use Newton's third law; equilibrium of forces on a particle and motion in a straight line (restricted to forces in two perpendicular directions or simple cases of forces given as 2-D vectors); application to problems involving smooth pulleys and connected particles; resolving forces in 2 dimensions; equilibrium of a particle under coplanar forces.

Assessed at AS and A-level

Teaching guidance

Connected particles

Students should:

- understand that usually strings will be modelled as light and inextensible
- understand that usually pulleys will be modelled as light and smooth
- understand that usually pegs will be modelled as smooth.

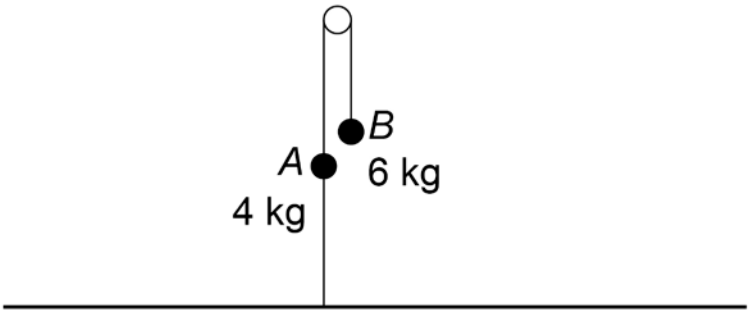
Notes

- Questions can be set involving objects that can be modelled as particles and are connected by a light, inextensible string.
- Questions can be set that involve contexts such as a car towing a trailer or several carriages connected together as a train.
- At AS, questions will be restricted to connected particles that move horizontally or vertically. Questions involving inclined planes will **not** be set.
- When particles are connected by a string so that they do not move in the same direction, the system must **not** be treated as moving with the same acceleration and the motion of each particle must be considered separately.

Examples

1 **In this question, use g = 10 m s^{-2}**

Two particles, A and B, have masses 4 kg and 6 kg respectively. They are connected by a light inextensible string that passes over a smooth fixed peg. A second light inextensible string is attached to A. The other end of the string is attached to the ground directly below A. The system remains at rest, as shown in the diagram.



(a) (i) Write down the tension in the string connecting A and B.

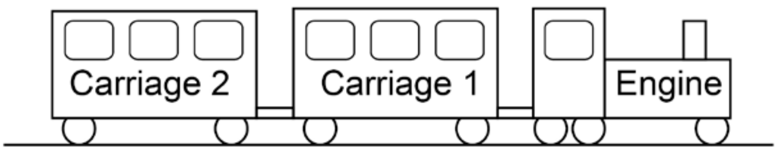
(ii) Find the tension in the string connecting A to the ground.

(iii) Find the magnitude of the total force exerted on the peg by the string connecting A and B.

(iv) Determine the magnitude and direction of the force exerted on the string connecting A and B by the peg.

(b) The string connecting particle A to the ground is cut. Find the acceleration of A after the string has been cut.

2 A small train at an amusement park consists of an engine and two carriages connected to each other by light horizontals rods, as shown in the diagram.



The engine has mass 2000 kg and each carriage has mass 500 kg.

The train moves along a straight horizontal track. A resistance force of magnitude 400 newtons acts on the engine, and resistance forces of magnitude 300 newtons act on each carriage. The train is accelerating at 0.5 m s^{-2}.

(a) Draw a diagram to show the **horizontal** forces acting on Carriage 2

(b) Show that the magnitude of the force that the rod exerts on Carriage 2 is 550 newtons.

(c) Find the magnitude of the force that rod attached to the engine exerts on Carriage 1

(d) A forward driving force of magnitude P newtons acts on the engine. Find P.

Only assessed at A-level

Teaching guidance

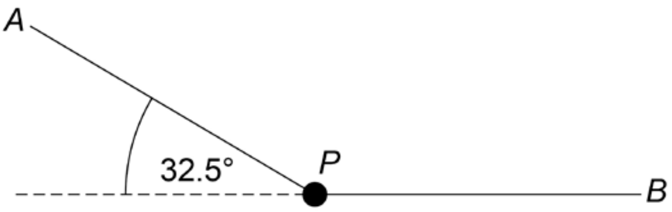
Students should be able to:

- answer questions that involve resolving forces
- understand that motion may not be restricted to horizontal or vertical and that inclined planes may be used.

Examples

1 **In this question, use $g = 9.81$ m s^{-2}**

A particle, of mass 10.5 kg, is suspended in equilibrium by two light strings, *AP* and *BP*. The string *AP* makes an angle of 32.5° to the horizontal and the other string, *BP*, is horizontal, as shown in the diagram.



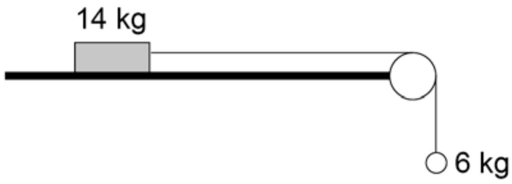
(a) Draw and label a diagram to show the forces acting on the particle.

(b) Show that the tension in the string *AP* is 192 N, correct to three significant figures.

(c) Find the tension in the horizontal string *BP*.

2 **In this question, use g = 9.8 m s^{-2}**

A block, of mass 14 kg, is held at rest on a rough horizontal surface. The coefficient of friction between the block and the surface is 0.25. A light inextensible string, which passes over a fixed smooth peg, is attached to the block. The other end of the string is attached to a particle, of mass 6 kg, which is hanging at rest.



The block is released and begins to accelerate.

(a) Find the magnitude of the friction force acting on the block.

(b) By forming two equations of motion, one for the block and one for the particle, show that the magnitude of the acceleration of the block and the particle is 1.2 m s^{-2}

(c) Find the tension in the string.

(d) When the block is released, it is 0.8 metres from the peg. Find the speed of the block when it hits the peg.

(e) When the block reaches the peg, the string breaks and the particle falls a further 0.5 metres to the ground. Find the speed of the particle when it hits the ground.

R5	Understand and use addition of forces; resultant forces; dynamics for motion in a plane.

Only assessed at A-level

Teaching guidance

Students should be able to:

- find resultants by use of a vector diagram or resolving into perpendicular components

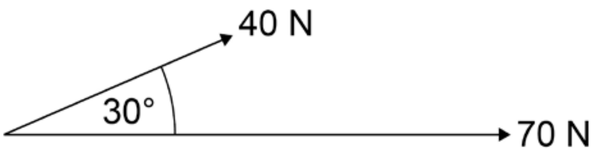
  Note: unless specifically stated in the question, any appropriate method for finding the resultant is acceptable.

- use $F = ma$ in the form $F = m\dfrac{\mathrm{d}v}{\mathrm{d}t}$ to set up and solve a differential equation.

  Note: mass will be constant.

Examples

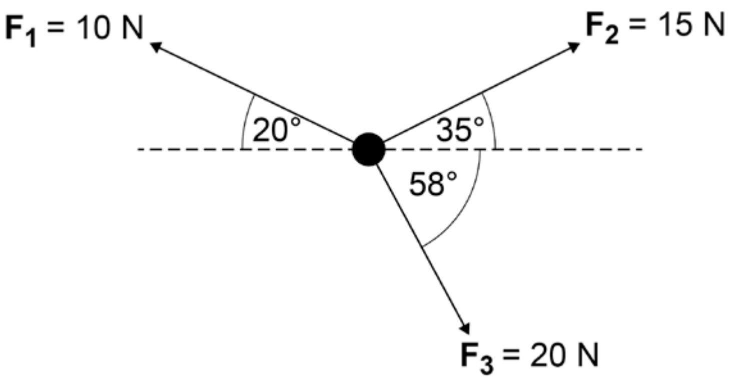
1 Two forces, acting at a point, have magnitudes of 40 newtons and 70 newtons. The angle between the two forces is 30°, as shown in the diagram.



(a) Find the magnitude of the resultant of these two forces.

(b) Find the angle between the resultant force and the 70 newton force.

2 Three forces $\mathbf{F}_1$, $\mathbf{F}_2$ and $\mathbf{F}_3$ act on a particle of mass 1.5 kg, as shown in the diagram.



Find the magnitude and direction of the acceleration of the particle.

3 **In this question use g = 9.8 m s^{-2}.**

Vicky has mass 65 kg and is skydiving. She steps out of a helicopter and falls vertically. She then waits a short period of time before opening her parachute. The parachute opens at time $t = 0$ when her speed is 19.6 m s^{-1}

She then experiences an air resistance force of magnitude 260v newtons, where v m s^{-1} is her speed at time t seconds.

(a) When $t > 0$

(i) Show that the resultant downward force acting on Vicky is 65 (9.8 – 4v) newtons.

(ii) Show that $\frac{\mathrm{d}v}{\mathrm{d}t} = -4(v - 2.45)$

(b) Find an expression for v in terms of t

R6

Understand and use the $F \leq \mu R$ model for friction; coefficient of friction; motion of a body on a rough surface; limiting friction and statics.

Only assessed at A-level

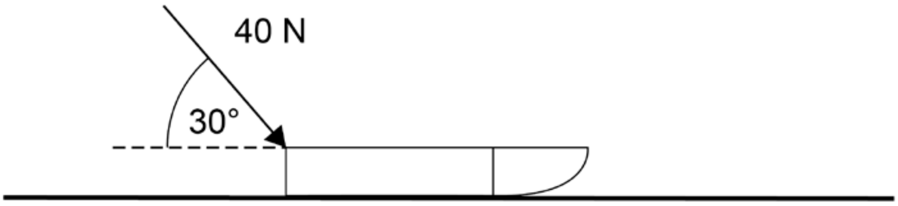
Teaching guidance

Students should be able to:

- answer questions set in the context of inclined planes
- understand that particles may be in equilibrium, limiting equilibrium or accelerating
- find the range of possible values for μ
- understand when $F = \mu R$ can be used.

1 **In this question use g = 9.8 m s^{-2}**

A sledge of mass 8 kg is at rest on a rough horizontal surface. A child tries to move the sledge by pushing it with a pole, as shown in the diagram, but the sledge **does not move**. The pole is at an angle of 30° to the horizontal and exerts a force of 40 newtons on the sledge.



(a) Draw a diagram to show the four forces acting on the sledge.

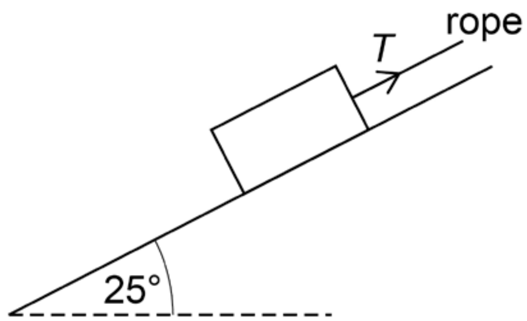
(b) Show that the normal reaction force between the sledge and the surface has magnitude 98 N.

(c) Find the magnitude of the friction force that acts on the sledge.

(d) Find the smallest possible value of the coefficient of friction between the sledge and the surface.

2 **In this question use g = 9.8 m s^{-2}**

A rough slope is inclined at an angle of 25° to the horizontal. A box of weight 80 newtons is on the slope. A rope is attached to the box and is parallel to the slope. The tension in the rope is of magnitude T newtons. The diagram shows the slope, the box and the rope.



(a) The box is held in equilibrium by the rope.

(i) Show that the normal reaction force between the box and the slope is 73 newtons.

(ii) The coefficient of friction between the box and the slope is 0.32. Find the magnitude of the maximum value of the frictional force which can act on the box.

(iii) Find the least possible tension in the rope to prevent the box from moving down the slope.

(iv) Find the greatest possible tension in the rope.

(v) Show that the mass of the box is approximately 8.2 kg.

(b) The rope is now released and the box slides down the slope. Find the acceleration of the box.

S Moments

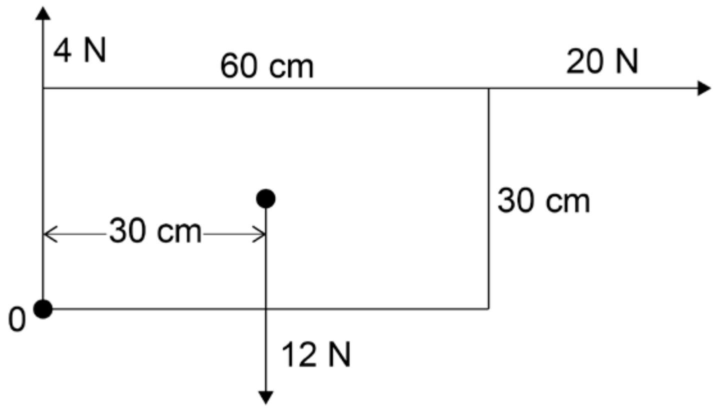
S1 Understand and use moments in simple static contexts.

Only assessed at A-level

Teaching guidance

Students should:

- use rods and laminae as appropriate models
- know that a rod or lamina will typically be uniform
- know that the centre of mass of uniform rods and rectangular laminae can be determined by symmetry
- be able to answer questions in which forces act in perpendicular directions. For example, to determine the resultant moment of a situation similar to the one below, about any given point:



Notes

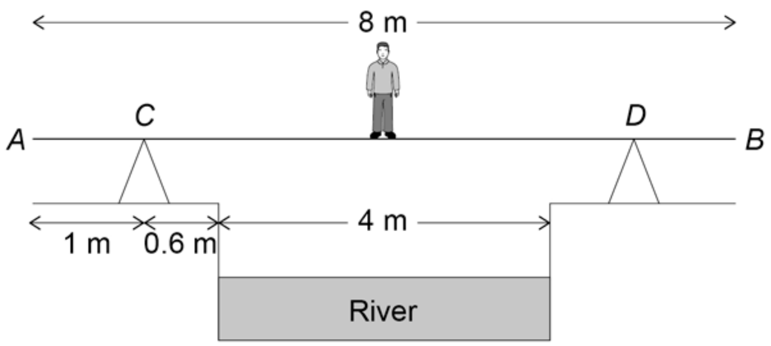
Forces will not need to be resolved into components.

Example

1 **In this question use $g = 9.8$ m s^{-2}**

Ken is trying to cross a river of width 4 m. He has a uniform plank, *AB*, of length 8 m and mass 17 kg. The ground on both edges of the river bank is horizontal. The plank rests at two points, *C* and *D*, on fixed supports which are on opposite sides of the river. The plank is at right angles to both river banks and is horizontal. The distance *AC* is 1 m, and the point *C* is at a horizontal distance 0.6 m from the river bank.

Ken, who has mass 65 kg, stands on the plank directly above the middle of the river, as shown in the diagram.



(a) Draw a diagram to show the forces acting on the plank.

(b) Given that the reaction on the plank at the point *D* is $44g$ N, where g is the acceleration due to gravity, find the horizontal distance of the point *D* from the nearest bank.

(c) State how you have used the fact that the plank is uniform in your solution.

2 **In this question use $g = 9.8$ m s^{-2}**

A pole, PQ, of length 3 metres and mass 32 kilograms rests horizontally in equilibrium on a support 0.75 metres from P and is held in place by a vertical string attached to the pole 0.5 metres from Q, as shown in the diagram.

(a) Show that the tension in the string is 130 N, to 2 significant figures.

(b) State an assumption you have made about the pole.

A force is now applied to the pole vertically downwards at Q.

The string holding the pole in place snaps when its tension exceeds 630 N.

(c) Find the maximum value of the force which can be applied at Q.

AQA

A1 Appendix A

Mathematical notation for AS and A-level qualifications in Mathematics

The tables below set out the notation that must be used by the AS and A-level Mathematics specification. Students will be expected to understand this notation without need for further explanation.

AS students will be expected to understand notation that relates to AS content, and will not be expected to understand notation that relates only to A-level content.

Set Notation		
1.1	$\in$	is an element of
1.2	$\notin$	is not an element of
1.3	$\subseteq$	is a subset of
1.4	$\subset$	is a proper subset of
1.5	$\{x_1, x_2, \ldots\}$	the set with elements x_1, x_2, …
1.6	$\{x : \ldots\}$	the set of all x such that …
1.7	$n(A)$	the number of elements in set A
1.8	$\varnothing$	the empty set
1.9	ε	the universal set
1.10	A'	the complement of the set A
1.11	$\mathbb{N}$	the set of natural numbers, $\{1, 2, 3, \ldots\}$
1.12	$\mathbb{Z}$	the set of integers, $\{0, \pm1, \pm2, \pm3, \ldots\}$
1.13	$\mathbb{Z}^+$	the set of positive integers, $\{1, 2, 3, \ldots\}$
1.14	$\mathbb{Z}_0^+$	the set of non-negative integers, $\{0, 1, 2, 3, \ldots\}$
1.15	$\mathbb{R}$	the set of real numbers

1.16	$\mathbb{Q}$	the set of rational numbers, $\left\{\frac{p}{q} : p \in \mathbb{Z}, q \in \mathbb{Z}^+\right\}$
1.17	$\cup$	union
1.18	$\cap$	intersection
1.19	(x, y)	the ordered pair x, y
1.20	$[a, b]$	the closed interval $\{x \in \mathbb{R} : a \le x \le b\}$
1.21	$[a, b)$	the interval $\{x \in \mathbb{R} : a \le x < b\}$
1.22	$(a, b]$	the interval $\{x \in \mathbb{R} : a < x \le b\}$
1.23	(a, b)	the open interval $\{x \in \mathbb{R} : a < x < b\}$

Miscellaneous symbols

2.1	$=$	is equal to
2.2	$\neq$	is not equal to
2.3	$\equiv$	is identical to or is congruent to
2.4	$\approx$	is approximately equal to
2.5	∞	infinity
2.6	$\propto$	is proportional to
2.7	$\therefore$	therefore
2.8	$\because$	because
2.9	$<$	is less than
2.10	$\leqslant$	is less than or equal to
2.11	$>$	is greater than
2.12	$\geqslant$	is greater than or equal to
2.13	$p \Rightarrow q$	p implies q (if p then q)
2.14	$p \Leftarrow q$	p is implied by q (if q then p)

2.15	$p \Leftrightarrow q$	p implies and is implied by q (p is equivalent to q)
2.16	a	first term of an arithmetic or geometric sequence
2.17	l	last term of an arithmetic sequence
2.18	d	common difference of an arithmetic sequence
2.19	r	common ratio of a geometric sequence
2.20	S_n	sum to n terms of a series
2.21	S_∞	sum to infinity of a geometric series

Operations

3.1	$a + b$	a plus b
3.2	$a - b$	a minus b
3.3	$a \times b$, ab, $a.b$	a multiplied by b
3.4	$a \div b$, $\frac{a}{b}$	a divided by b
3.5	$\sum_{i=1}^{n} a_i$	$a_1 + a_2 + \ldots a_n$
3.6	$\prod_{i=1}^{n} a_i$	$a_1 \times a_2 \times \ldots a_n$ This notation is not required in AS or A-level Mathematics
3.7	$\sqrt{a}$	the non-negative square root of a
3.8	$\|a\|$	the modulus of a
3.9	$n!$	n factorial: $n! = n \times (n-1) \times \ldots \times 2 \times 1,\ n \in \mathbb{N}$; $0! = 1$
3.10	$\binom{n}{r}$, ${}^{n}\mathrm{C}_{r}$, ${}_{n}\mathrm{C}_{r}$	the binomial coefficient $\frac{n!}{r!(n-r)!}$ for $n, r \in \mathbb{Z}_0^+, r \le n$ or $\frac{n(n-1)\ldots(n-r+1)}{r!}$ for $n \in \mathbb{Q}, r \in \mathbb{Z}_0^+$

Functions

4.1	$\mathrm{f}(x)$	the value of the function f at x
4.2	$\mathrm{f} : x \mapsto y$	the function f maps the element x to the element y
4.3	f^{-1}	the inverse function of the function f
4.4	gf	the composite function of f and g which is defined by $\mathrm{gf}(x) = \mathrm{g}(\mathrm{f}(x))$
4.5	$\lim_{x \to a} \mathrm{f}(x)$	the limit of $\mathrm{f}(x)$ as x tends to a
4.6	Δx , δx	an increment of x
4.7	$\frac{\mathrm{d}y}{\mathrm{d}x}$	the derivative of y with respect to x
4.8	$\frac{\mathrm{d}^n y}{\mathrm{d}x^n}$	the nth derivative of y with respect to x
4.9	$\mathrm{f}'(x), \mathrm{f}''(x), \ldots, \mathrm{f}^{(n)}(x)$	the first, second, …, nth derivatives of $\mathrm{f}(x)$ with respect to x
4.10	$\dot{x}, \ddot{x}, \ldots$	the first, second, … derivatives of x with respect to t This notation will not be used in AS and A-level Mathematics
4.11	$\int y \,\mathrm{d}x$	the indefinite integral of y with respect to x
4.12	$\int_a^b y \,\mathrm{d}x$	the definite integral of y with respect to x between the limits $x = a$ and $x = b$

Exponentials and logarithmic functions

5.1	e	base of natural logarithms
5.2	e^x, $\exp x$	exponential function of x The notation $\exp x$ will not be used in this specification.
5.3	$\log_a x$	logarithm to the base a of x
5.4	$\ln x$, $\log_e x$	natural logarithm of x

Trigonometric functions

6.1	$\left.\begin{array}{l}\sin, \cos, \tan \\ \operatorname{cosec}, \sec, \cot\end{array}\right\}$	the trigonometric functions
6.2	$\left.\begin{array}{l}\sin^{-1}, \cos^{-1}, \tan^{-1} \\ \arcsin, \arccos, \arctan\end{array}\right\}$	the inverse trigonometric functions
6.3	$^\circ$	degrees
6.4	rad	radians Typically no symbol will be used to denote the use of radian measure.

Vectors

9.1	$\mathbf{a}$, $\underline{\mathbf{a}}$, $\underset{\sim}{\mathbf{a}}$	the vector $\mathbf{a}$, $\underline{\mathbf{a}}$, $\underset{\sim}{\mathbf{a}}$; these alternatives apply throughout section 9
9.2	$\overrightarrow{AB}$	the vector represented in magnitude and direction by the directed line segment AB
9.3	$\hat{\mathbf{a}}$	a unit vector in the direction of $\mathbf{a}$
9.4	$\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$	unit vectors in the directions of the Cartesian coordinate axes
9.5	$\lvert\mathbf{a}\rvert$, a	the magnitude of $\mathbf{a}$
9.6	$\lvert\overrightarrow{AB}\rvert$, AB	the magnitude of $\overrightarrow{AB}$
9.7	$\begin{bmatrix} a \\ b \end{bmatrix}$, $a\mathbf{i}+b\mathbf{j}$	column vector and corresponding unit vector notation We will use square brackets, but round brackets are perfectly acceptable.
9.8	$\mathbf{r}$	position vector
9.9	$\mathbf{s}$	displacement vector
9.10	$\mathbf{v}$	velocity vector
9.11	$\mathbf{a}$	acceleration vector

Probability and statistics		
11.1	A, B, C, etc	events
11.2	$A \cup B$	union of the events A and B
11.3	$A \cap B$	intersection of the events A and B
11.4	$\mathrm{P}(A)$	probability of the event A
11.5	A'	complement of the event A
11.6	$\mathrm{P}(A \mid B)$	probability of the event A conditional on the event B
11.7	X, Y, R, etc	random variables
11.8	x, y, r, etc	values of the random variables X, Y, R, etc
11.9	$x_1, x_2, \ldots$	values of observations
11.10	$f_1, f_2, \ldots$	frequencies with which the observations $x_1, x_2, \ldots$occur
11.11	$\mathrm{p}(x)$, $\mathrm{P}(X = x)$	probability function of the discrete random variable X
11.12	$\mathrm{p}_1, \mathrm{p}_2, \ldots$	probabilities of the values $x_1, x_2, \ldots$ of the discrete random variable X
11.13	$\mathrm{E}(X)$	expectation of the random variable X Not required in AS and A-level Mathematics.
11.14	$\mathrm{Var}(X)$	variance of the random variable X Not required in AS and A-level Mathematics.
11.15	$\sim$	has the distribution
11.16	$\mathrm{B}(n, p)$	binomial distribution with parameters n and p, where n is the number of trials and p is the probability of success in a trial
11.17	q	$q = 1 - p$ for binomial distribution
11.18	$\mathrm{N}(\mu, \sigma^2)$	Normal distribution with mean μ and variance σ^2
11.19	$Z \sim \mathrm{N}(0,1)$	standard Normal distribution
11.20	ϕ	probability density function of the standardised Normal variable with distribution $N(0,1)$ Not required in AS and A-level Mathematics.

11.21	Φ	corresponding cumulative distribution function Not required in AS and A-level Mathematics.
11.22	μ	population mean
11.23	σ^2	population variance
11.24	σ	population standard deviation
11.25	$\bar{x}$	sample mean
11.26	s^2	sample variance Not required in AS and A-level Mathematics.
11.27	s	sample standard deviation Not required in AS and A-level Mathematics.
11.28	H_0	null hypothesis
11.29	H_1	alternative hypothesis
11.30	r	product moment correlation coefficient for a sample
11.31	ρ	product moment correlation coefficient for a population

Mechanics		
12.1	kg	kilogram(s)
12.2	m	metre(s)
12.3	km	kilometre(s)
12.4	m/s, m s^{-1}	metre(s) per second (velocity)
12.5	m/s^2, m s^{-2}	metre(s) per second per second (acceleration)
12.6	F	force or resultant force
12.7	N	newton
12.8	Nm	newton metre (moment of force)
12.9	t	time
12.10	s	displacement
12.11	u	initial velocity
12.12	v	velocity or final velocity
12.13	a	acceleration
12.14	g	acceleration due to gravity
12.15	μ	coefficient of friction

A2 Appendix B

Mathematical formulae and identities

Students must use the following formulae and identities for AS and A-level Mathematics, without these formulae and identities being provided, either in these forms or in equivalent forms. These formulae and identities may only be provided where they are the starting point for a proof or as a result to be proved.

Pure mathematics	
Quadratic equations	$ax^2 + bx + c = 0$ has roots $\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Laws of indices	$a^x a^y \equiv a^{x+y}$ $a^x \div a^y \equiv a^{x-y}$ $(a^x)^y \equiv a^{xy}$
Laws of logarithms	$x = a^n \Leftrightarrow n = \log_a x$ for $a > 0$ and $x > 0$ $\log_a x + \log_a y \equiv \log_a (xy)$ $\log_a x - \log_a y \equiv \log_a \left(\dfrac{x}{y}\right)$ $k \log_a x \equiv \log_a (x^k)$
Coordinate geometry	A straight line, gradient m passing through (x_1, y_1) has equation $y - y_1 = m(x - x_1)$ Straight lines with gradients m_1 and m_2 are perpendicular when $m_1 m_2 = -1$
Sequences	General term of an arithmetic progression: $u_n = a + (n - 1)d$ General term of a geometric progression: $u_n = ar^{n-1}$

Trigonometry	In the triangle ABC: sine rule: $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$ cosine rule: $a^2 = b^2 + c^2 - 2bc\cos A$ area: $\dfrac{1}{2}ab\sin C$ $\cos^2 A + \sin^2 A \equiv 1$ $\sec^2 A \equiv 1 + \tan^2 A$ $\text{cosec}^2 A \equiv 1 + \cot^2 A$ $\sin 2A \equiv 2\sin A\cos A$ $\cos 2A \equiv \cos^2 A - \sin^2 A$ $\tan 2A \equiv \dfrac{2\tan A}{1 - \tan^2 A}$
Mensuration	Circumference (C) and area (A) of a circle, radius r and diameter d. $C = 2\pi r = \pi d$ $A = \pi r^2$ Pythagoras' Theorem: In any right-angled triangle, where a, b and c are the lengths of the sides and c is the hypotenuse: $c^2 = a^2 + b^2$ Area of a trapezium: $\dfrac{1}{2}(a+b)h$ where a and b are the lengths of the parallel sides and h is their perpendicular separation Volume of a prism $=$ area of cross section $\times$ length For a circle of radius r, where an angle at the centre of θ radians subtends an arc of length s and encloses an associated sector of area A: $s = r\theta$ $A = \dfrac{1}{2}r^2\theta$

Calculus and differential equations

Differentiation

Function	Derivative
x^n	nx^{n-1}
$\sin kx$	$k \cos kx$
$\cos kx$	$-k \sin kx$
e^{kx}	ke^{kx}
$\ln x$	$\frac{1}{x}$
$f(x) + g(x)$	$f'(x) + g'(x)$
$f(x)g(x)$	$f'(x)g(x) + f(x)g'(x)$
$f(g(x))$	$f'(g(x))g'(x)$

Integration

Function	Derivative
x^n	$\frac{1}{n+1}x^{n+1} + c, n \neq -1$
$\cos kx$	$\frac{1}{k}\sin kx + c$
$\sin kx$	$-\frac{1}{k}\cos kx + c$
e^{kx}	$\frac{1}{k}e^{kx} + c$
$\frac{1}{x}$	$\ln\lvert x\rvert + c, x \neq 0$ Modulus notation is not required
$f'(x) + g'(x)$	$f(x) + g(x) + c$
$f'(g(x))g'(x)$	$f(g(x)) + c$

Area under a curve $= \int_a^b y \, dx \; (y \geq 0)$

Vectors	$\lvert x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \rvert = \sqrt{(x^2 + y^2 + z^2)}$

Mechanics	
Forces and equilibrium	Weight = mass x g Friction: $F \le \mu R$ Newton's second law: $F = ma$
Kinematics	For motion in a straight line with variable acceleration: $v = \frac{\mathrm{d}r}{\mathrm{d}t}$ $a = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}^2 r}{\mathrm{d}t^2}$ $r = \int v \, \mathrm{d}t$ $v = \int a \, \mathrm{d}t$

Statistics	
The mean of a set of data	$\bar{x} = \frac{\sum x}{n} = \frac{\sum fx}{\sum f}$
The standard Normal variable	$Z = \frac{X - \mu}{\sigma}$ where $X \sim \mathrm{N}\left(\mu, \sigma^2\right)$

Get help and support

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aqa.org.uk

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